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Does dynamic market competition with technological innovation leave no one behind?: Examinations of Hicksian optimism in OLG frameworks*

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Abstract

In a simple Overlapping Generations model, we define ‘improvements’ due to technological progress in the economy and introduce and study the notion of Hicksian optimism, which captures the idea that no one, in the ‘long-run’, is left behind due to such improvements in the economy, and show that the market mechanism with technological progress unavoidably leaves some individuals behind and cannot deliver the Hicksian optimism. This negative result holds for a broader class of intertemporal resource allocation mechanisms.

Keywords: dynamic market competition with technological progress, Hicksian optimism, Walrasian allocation rule, weak Pareto efficiency, individual rationality, overlapping generations economy

JEL Classification Numbers: D30, D51, D60, O33, P10, P20, P40

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1 Introduction

The market mechanism is central to the modern economy and is believed widely by economists to be crucial in improving individual welfare progressively through its dynamic process of competition among the producers¹ as such competition promotes technical progress and enhances the productivity of the economy as a whole.

Economic research on whether or not market competition is the best vehicle to promote technological innovation is vast. The recent literature on the topic has illustrated a complicated picture between competition and innovation (see, among others, Aghion et al. (2005), Igami and Uetake (2020)). However, even setting aside the problem of promoting technological innovation by market competition, the question that the market mechanism can ensure the progressive improvement of individual welfare remains. This is because the introduction of a new technology often involves a radical change of economic structure, which leads to the division of the population into the “winners” and the “losers”.²

The issue concerning the winners and losers seems to be easily dealt with by the Kaldor-Hicks compensation test (Hicks (1939), Kaldor (1939)). Indeed, if a technical progress due to innovation can be formulated as an expansion of the production possibility set available in the economy à la Schumpeter (1942), then a policy encouraging such innovations would always be socially desirable with respect to the (weaker) Kaldor-Hicks hypothetical compensation principle. This is because, a change from a competitive equilibrium under one economy to another competitive equilibrium under possibly a different economy involves an *expansion* of the production possibility set in terms of set inclusion, and consequently, the aggregate sum of all consumers’ ‘upper contour sets’ definitely has a non-empty intersection with the new production possibility set. Therefore, even if some consumer becomes a “loser” with this change, this consumer can be (hypothetically) compensated via a suitable shift of an aggregate supply of commodity bundles, and a suitable redistribution of the goods.

Such a compensation is, however, just hypothetical and the Kaldor-Hicks compensation principle provides no mechanism to implement such a compensation, and as a result, a loser would be left as a “loser” in reality. Therefore, judging such possible improvements of individual welfare by means of the Kaldor-Hicks compensation principle alone would not be much help if the hypothetical compensation cannot be actually implemented.

Nevertheless, neoclassical economics has attempted to justify the application of the (somewhat modified) hypothetical compensation principle to evaluate possible improvement of social welfare by appealing to the so-called *Hicksian optimism*. As stated by Corden (1984, page 68), “if Pareto efficient policies are being pursued consistently over a long period, the *chances* are that eventually—though not at every particular step—everyone will be better off.” He labels this

¹Producers are motivated by capturing extra profits via the access to the most advanced technology.

²See Frey (2019) for a fascinating historical account of the consequences of the introduction of new technologies.

principle as the *Hicksian optimism* and attributes it to Hicks (1941, p. 111) where Hicks writes:

“... then, although we could not say that all the inhabitants of that community would be necessarily better off than they would have been if the community had been organized on some different principle, nevertheless there would be a strong probability that almost all of them would be better off after the lapse of a sufficient length of time. Substantially, that is the creed of classical economics; if the ‘improvements’ are properly defined, it would appear to be a creed that is soundly based.”

The meaning of “Pareto efficient policies” mentioned by Corden (1984) above is “policy shifts in the direction of improved Pareto efficient rankings”. In other words, a Pareto efficient policy is to shift the utility possibility frontier of society in a ‘north-east’ direction. Typical examples of such policies would be to improve productive efficiency, such as reorganizations of production or promoting R&D investments, as Hicks (1941) mentioned.

This type of policy shift meets the Kaldor-Hicks compensation principle. However, this policy also involves moving from one Pareto-efficient allocation located on the original utility possibility frontier to another Pareto-efficient allocation located on the more ‘north-eastern’ utility possibility frontier. This movement will likely make some people worse off while making others better off. The Pareto principle itself cannot be used for making welfare judgments in such a movement. This is where the Hicksian optimism fills in.

The Hicksian optimism presumes that such policies are consistently applied over a long period, so that they lead gradually from a more ‘*south-western*’ utility possibility frontier to a more ‘*north-eastern*’ utility possibility frontier. It then claims that after a sufficient length of time, the Pareto-efficient allocation that is eventually reached on the most ‘north-eastern’ utility possibility frontier would be Pareto-superior to the original Pareto-efficient allocation.

Therefore, if we follow through the above idea of the Hicksian optimism, we may conjecture that the innovation and technological progress in the market place can eventually benefit every individual in the society. It may be noted that Kandori (2023, p. 367) presents a slightly weaker version of the Hicksian optimism by observing that “... the experiences of the twentieth century taught us that, if we *adopt the compensation principle and apply it consistently, many people will be better off in the long run.*” In a way, the Hicksian optimism provides a foundation for achieving some goals like “inclusive economic growth”, “shared prosperity”, and “no one being left behind” envisioned and set forth in the 2030 Agenda for Sustainable Development as pledged by 193 United Nations Member States.³

The purpose of this paper is to examine the above conjecture. To do so, we study a *simple overlapping generations economy*. In this economy, time is discrete and infinite. Each generation consists of a single individual who lives

³See <https://sdgs.un.org/2030agenda>.

through two periods: working while young and retiring (and then dying) while old. Each generation has a well-behaved utility function for lifetime consumption activities and is endowed with one unit of labor at young age. There is a convex production possibility set in each period that can vary over time, incorporating the “improvements” that Hicks had in mind. We introduce and formulate the notion of Hicksian optimism in this context.

To do so, we define an *intertemporal allocation rule* that specifies a subset of desirable intertemporal allocations feasible for each simple overlapping generations economy. A prominent example of such an allocation rule is the *Walrasian intertemporal allocation rule*, which specifies the set of Walrasian competitive equilibrium allocations for each and every period of a simple overlapping generations economy.

The notion of *Hicksian optimism* is then formulated as follows: an intertemporal allocation rule should select a feasible intertemporal allocation that will *eventually* improve the welfare of generations in subsequent periods compared to those in the initial period, provided that the simple overlapping generations economy grows over time due to innovation and technological progress and then transitions to a stationary overlapping generations economy.

To formulate the notion of *Hicksian optimism* as axioms, we introduce two approaches to measuring the welfare of generations in each period. In each period, the welfare of a generation at old age is definitely measured by the utility of her consumption at that age. In contrast, it could be debatable to identify the welfare of a generation at young age, and there are at least two ways to measure it. According to Hicks’s original idea of optimism, which focuses on comparing each period’s temporal welfare allocation, the welfare of a generation at young age should be determined by her *temporal utility* at that age. However, one could argue that the welfare of such an individual should be measured by her *lifetime utility* rather than her utility at young age, since she will live not only in her initial period but also in a subsequent period. Thus, she will have the opportunity to compensate for her disadvantageous welfare in her youth with her advantageous welfare in her old age.

Based on the aforementioned two views, we introduce two variants of the axioms of *Hicksian optimism*: *Hicksian Optimism (HO)* and *Hicksian Optimism with Lifetime Welfare (HOLW)*. Regardless of which variant is applied, we demonstrate that an efficient, individually rational intertemporal allocation rule cannot satisfy the axioms of *Hicksian optimism*. As a corollary, the Walrasian intertemporal allocation rule is also shown to violate the axioms of *Hicksian optimism*, implying that the market mechanism cannot eventually benefit all citizens via expansions of society’s production possibility sets due to innovation and technological progress.

It may be noted that an overlapping generations model can realistically capture the real world where multiple individuals with different one-period utility functions co-exist in each period and each individual survives for a finite period of time only. It may be noted further that resource transfers between generations across time periods can unfold endogenously. Thus, overlapping generations models can address the problem of both contemporaneous resource

allocation among individuals with different preferences within the same period and resource allocation across time periods (i.e., the allocation between current consumption and capital accumulation). These features of overlapping generations models are thus well-suited for testing Hicksian optimism since it focuses on how contemporaneous welfare allocation among multiple coexisting individuals with different one-period utility functions changes over successive time periods within the context of intertemporal resource allocations.

The remainder of the paper is organized as follows. Section 2 presents our basic model. Section 3 formulates the notion of *Hicksian optimism*. Section 4 examines whether any efficient and individually rational intertemporal allocation rule can satisfy each of the two axioms of *Hicksian optimism*. We conclude in Section 5.

2 A Basic Model

Time is discrete $t = 1, 2, \dots$ and let the time periods be indexed by $\mathcal{T} = \{1, 2, \dots\}$. Consider a simple Overlapping Generations (OLG) economy in which each generation $t = 1, 2, \dots$, is a single individual, lives for two periods, works only at young and retires at old and then dies. In the initial period $t = 1$, there are two individuals: one is the generation $t = 0$ at the old age (and dies at the end of the period $t = 1$) and the other is generation $t = 1$ at the young age. There are $n \geq 2$ commodities that are produced in this economy and can be used as consumption goods and/or capital goods.

Each generation $t = 1, 2, \dots$, at the young age is endowed with one unit of labor endowment, $\omega_l = 1$. The generation $t = 0$ at the old age is endowed with a commodity vector $\omega_a^0 \in \mathbb{R}_+^n$.⁴ Let ω_a^0 be the saving of the generation $t = 0$ at the young age in period $t = 0$ (outside this model analysis). For each generation $t = 1, 2, \dots$, $x_b^t \in \mathbb{R}_+^n$ represents a consumption bundle that is consumed by generation t at young and $x_a^t \in \mathbb{R}_+^n$ represents a consumption bundle that is consumed by the generation t at old. We shall use $\omega_a^t \in \mathbb{R}_+^n$ to denote a commodity bundle that is saved by the generation t at young for the sake of the consumption activity at old. $l^t \in [0, 1]$ denotes the amount of labor supplied by the generation t at young.

Each generation $t = 0, 1, 2, \dots$, has a common *utility function of lifetime consumption activities* as follows: $u : \mathbb{R}_+^n \times \mathbb{R}_+^n \rightarrow \mathbb{R}$ such that there are two *one-period utility functions*, $v_b : \mathbb{R}_+^n \rightarrow \mathbb{R}$ and $v_a : \mathbb{R}_+^n \rightarrow \mathbb{R}$, and a real-valued function $U : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ satisfying $u(x_b, x_a) = U(v_b(x_b), v_a(x_a))$ for all $(x_b, x_a) \in \mathbb{R}_+^n \times \mathbb{R}_+^n$. We assume that $U(\cdot, \cdot)$ is continuous and is increasing in each of its arguments, and, for $i = a, b$, v_i is continuous, quasi-concave and increasing on $\mathbb{R}_+^n$.⁵

⁴For any positive integer m , $\mathbb{R}^m$ (resp. $\mathbb{R}_+^m$, $\mathbb{R}_{++}^m$ and $\mathbb{R}_-^m$) denotes the m -fold Cartesian product of $\mathbb{R} = (-\infty, +\infty)$ (resp. $\mathbb{R}_+ = [0, +\infty)$, $\mathbb{R}_{++} = (0, +\infty)$ and $\mathbb{R}_- = (-\infty, 0]$). For any $a, b \in \mathbb{R}^m$, $a \geq b$ denotes $[a_1 \geq b_1, \dots, a_m \geq b_m]$, $a > b$ denotes $[a \geq b \text{ and } a \neq b]$, and $a > b$ denotes $[a_1 > b_1, \dots, a_m > b_m]$.

⁵ v_i is increasing if for all x_i and x'_i with $x_i \geq x'_i$, we have $v_i(x_i) \geq v_i(x'_i)$ and $v_i(x_i) > v_i(x'_i)$

In each period $t \in \mathcal{T}$, there is a production possibility set $Y^t \subseteq \mathbb{R}_- \times \mathbb{R}_+^n \times \mathbb{R}_+^n$ associated with a generic element $y \equiv (-y_l, -\underline{y}, \bar{y}) \in Y^t$, where $y_l \in \mathbb{R}_+$ represents a direct labor input, $\underline{y} \in \mathbb{R}_+^n$ represents a vector of commodity inputs, and $\bar{y} \in \mathbb{R}_+^n$ represents a vector of gross outputs. Assume that Y^t is a closed convex cone with $\mathbf{0} \in Y^t$ and satisfies the following properties:

- A1.** There are two subsets Y_b^t and $\underline{Y}^t$ of Y^t such that⁶ $\text{co}(Y_b^t \cup \underline{Y}^t) = Y^t$ and
- (i) for any $y = (-y_l, -\underline{y}, \bar{y}) \in \underline{Y}^t$, if $\bar{y} \geq \mathbf{0}$, then $\underline{y} \geq \mathbf{0}$ and $y_l > 0$;
 - (ii) $Y_b^t \equiv \cup_{\lambda \in \mathbb{R}_+} \{\lambda(-1, \mathbf{0}, \omega_b^t)\}$ for some $\omega_b^t \in \mathbb{R}_+^n$ such that, for some $y' = (-1, -\underline{y}', \bar{y}') \in \underline{Y}^t$, $\bar{y}' - \underline{y}' \geq \omega_b^t$ holds,
- A2.** For any $d \in \mathbb{R}_+^n$, there exists $y = (-y_l, -\underline{y}, \bar{y}) \in Y^t$ such that $\bar{y} - \underline{y} \geq d$,
- A3.** For any $y = (-y_l, -\underline{y}, \bar{y}) \in Y^t$ and any $(-y'_l, -\underline{y}') \in \mathbb{R}_- \times \mathbb{R}_+^n$, if $(-y'_l, -\underline{y}') \leq (-y_l, -\underline{y})$, then $(-y'_l, -\underline{y}', \bar{y}) \in Y^t$.

A3 is the standard property of *free disposal*. **A2** represents the standard property of *productiveness* and requires that any final demand vector can be produced as a net output vector via some production activities. **A1** implies that there are two types of production techniques: one type, represented by Y_b^t , is perfectly labor intensive in which a non-negative vector of gross outputs can be produced only by labor inputs, and the other type, represented by $\underline{Y}^t$, requires both labor input and non-labor inputs to produce a non-zero and non-negative vector of gross outputs. **A1** further requires that the net output productivity of the perfectly labor intensive technique Y_b^t is *inferior to* the technique contained in $\underline{Y}^t$: the net output vector ω_b^t produced by one unit of labor in Y_b^t is dominated by a net output vector produced with one unit of labor input in $\underline{Y}^t$. It may be noted that the generation t at the young age can produce ω_b^t by using one unit of the labor endowment and then allocate it for consumption at young and old ages in an autarkic economy.

An *economy* in period $t \in \mathcal{T}$, to be denoted by E_t , is a triple $\langle Y^t; \omega_l; u \rangle$. Note that in the economy E_t , there are two individuals, the generation $t-1$ at the old age and the generation t at the young age. Let $\mathcal{E}^t$ be the set of all available economies in period $t \in \mathcal{T}$.

For an economy $E_t \in \mathcal{E}^t$ associated with the savings bundle $\omega_a^{t-1} \in \mathbb{R}_+^n$ of generation $t-1$ at the old age, a profile $(x_a^{t-1}; (x_b^t, x_a^t), l^t, \omega_a^t; (-y_l^t, -\underline{y}^t, \bar{y}^t))$ is called a *temporary allocation in period t* if and only if $x_a^{t-1} \in \mathbb{R}_+^n$; $(x_b^t, x_a^t) \in \mathbb{R}_+^n \times \mathbb{R}_+^n$, $l^t \in [0, 1]$, and $\omega_a^t \in \mathbb{R}_+^n$; and $(-y_l^t, -\underline{y}^t, \bar{y}^t) \in Y^t$. Moreover, a temporary allocation $(x_a^{t-1}; (x_b^t, x_a^t), l^t, \omega_a^t; (-y_l^t, -\underline{y}^t, \bar{y}^t))$ is *feasible in period t* if and only if $x_a^{t-1} + x_b^t + \omega_a^t \leq \bar{y}^t - \underline{y}^t + \omega_a^{t-1}$ with $\underline{y}^t \leq \omega_a^{t-1}$ and $y_l^t \leq l^t$. Denote the set of temporary feasible allocations for the economy E_t with ω_a^{t-1} by $F(E_t; \omega_a^{t-1})$.

Define a *general OLG economy* by $(E_t = \langle Y^t; \omega_l; u \rangle)_{t=1}^\infty$. Let $\mathcal{E}^\infty$ be the set of all general OLG economies with generic element $E^\infty = (E_1, \dots, E_t, \dots)$.

for all x_i and x'_i with $x_i > x'_i$.

⁶For any set $K \subseteq \mathbb{R}^k$, $\text{co}(K)$ denotes the *convex hull* of K .

Given a general OLG economy $E^\infty = (E_1, \dots, E_t, \dots) \in \mathcal{E}^\infty$ associated with the savings bundle $\omega_a^0 \in \mathbb{R}_+^n$ of generation $t = 0$ at the old age, a profile

$$\left(x_a^0; ((x_b^t, x_a^t), l^t, \omega_a^t)_{t \in \mathcal{T}}; (-y_l^t, -\underline{y}^t, \bar{y}^t)_{t \in \mathcal{T}} \right) \in \mathbb{R}_+^n \times \prod_{t=1}^{\infty} (\mathbb{R}_+^n \times \mathbb{R}_+^n \times [0, 1] \times \mathbb{R}_+^n \times Y^t),$$

is called an *intertemporal allocation* if and only if for each period $t \in \mathcal{T}$, $(x_a^{t-1}; (x_b^t, x_a^t), l^t, \omega_a^t; (-y_l^t, -\underline{y}^t, \bar{y}^t))$ is a temporary allocation in period t associated with the saving bundle $\omega_a^{t-1} \in \mathbb{R}_+^n$ of generation $t - 1$ at the old age. Moreover, given a general OLG economy $E^\infty = (E_1, \dots, E_t, \dots) \in \mathcal{E}^\infty$ associated with the savings bundle $\omega_a^0 \in \mathbb{R}_+^n$ of generation $t = 0$ at the old age, an intertemporal allocation $\left(x_a^0; ((x_b^t, x_a^t), l^t, \omega_a^t)_{t \in \mathcal{T}}; (-y_l^t, -\underline{y}^t, \bar{y}^t)_{t \in \mathcal{T}} \right)$ is *feasible for the economy E^∞ with $\omega_a^0 \in \mathbb{R}_+^n$* if and only if

$$\begin{aligned} x_a^0 + x_b^1 + \omega_a^1 &\leq \bar{y}^1 - \underline{y}^1 + \omega_a^0 \text{ with } \underline{y}^1 \leq \omega_a^0 \text{ and } y_l^1 \leq l^1; \\ \text{and } x_a^{t-1} + x_b^t + \omega_a^t &\leq \bar{y}^t - \underline{y}^t + \omega_a^{t-1} \text{ with } \underline{y}^t \leq \omega_a^{t-1} \text{ and } y_l^t \leq l^t \quad (\forall t \in \mathcal{T}). \end{aligned}$$

Denote the set of intertemporal feasible allocations for the economy E^∞ with $\omega_a^0 \in \mathbb{R}_+^n$ by $F(E^\infty; \omega_a^0)$.

Consider a mapping φ of $\mathcal{E}^\infty \times \mathbb{R}_+^n$ into a universal set of intertemporal allocations. This mapping is called an *intertemporal allocation rule* if and only if for each $E^\infty \in \mathcal{E}^\infty$ associated with $\omega_a^0 \in \mathbb{R}_+^n$, $\emptyset \neq \varphi(E^\infty; \omega_a^0) \subseteq F(E^\infty; \omega_a^0)$ holds. In the following discussion, as a slight abuse of notation for simplicity, let us simply denote $\varphi(E^\infty)$ instead of $\varphi(E^\infty; \omega_a^0)$.

The market mechanism is a prominent example of an (intertemporal) allocation rule: it selects Walrasian competitive equilibrium allocations for each general OLG economy E^∞ associated with $\omega_a^0 \in \mathbb{R}_+^n$. Given a general OLG economy E^∞ associated with $\omega_a^0 \in \mathbb{R}_+^n$, a sequence of commodity prices, rental prices of capital service, and wages, $\{(\mathbf{p}_{t-1}, \mathbf{q}_t, w_t)\}_{t=1}^\infty$ with $\mathbf{p}_{t-1} \in \mathbb{R}_+^n$, $\mathbf{q}_t \in \mathbb{R}_+^n$, and $w_t \in \mathbb{R}_+$ ($\forall t \in \mathcal{T}$), and an intertemporal allocation $\left(x_a^0; ((x_b^t, x_a^t), l^t, \omega_a^t)_{t \in \mathcal{T}}; (-y_l^t, -\underline{y}^t, \bar{y}^t)_{t \in \mathcal{T}} \right)$ constitute a *Walrasian competitive equilibrium for E^∞* if and only if the following conditions are satisfied:

(i) for each $t \in \mathcal{T}$, $(-y_l^t, -\underline{y}^t, \bar{y}^t)$ is a solution to the following profit maximization problem:

$$\max_{(-y_l^t, -\underline{y}^t, \bar{y}^t) \in Y^t} \mathbf{p}_t \bar{y}^t - \mathbf{q}_t \underline{y}^t - w_t y_l^t;$$

(ii) for each generation $t = 1, 2, \dots$, $((x_b^t, x_a^t), l^t, \omega_a^t)$ is a solution to the following utility maximization problem:

$$\max_{((x_b, x_a), l, \omega_a)} u(x_b, x_a)$$

subject to:

$$\begin{aligned} \mathbf{p}_t x_b + \mathbf{p}_t \omega_a &\leq w_t l; \\ \mathbf{p}_t \underline{y}^{t+1} &\leq \mathbf{p}_t \omega_a; \\ \mathbf{p}_{t+1} x_a &\leq \mathbf{q}_{t+1} \underline{y}^{t+1} + \mathbf{p}_t \cdot (\omega_a - \underline{y}^{t+1}) + \pi^{t+1}; \\ l &\in [0, 1] \text{ and } \omega_a \in \mathbb{R}_+^m; \end{aligned}$$

where

$$\pi^t \equiv \max_{(-y_l^t, -\underline{y}^t, \bar{y}^t) \in Y^t} \mathbf{p}_t \bar{y}^t - \mathbf{q}_t \underline{y}^t - w_t y_l^t \quad (\forall t \in \mathcal{T});$$

(iii) for generation $t = 0$, x_a^0 is a solution to the following utility maximization problem:

$$\max v_a(x_a)$$

subject to:

$$\begin{aligned} \mathbf{p}_0 \underline{y}^1 &\leq \mathbf{p}_0 \omega_a^0; \\ \mathbf{p}_1 x_a &\leq \mathbf{q}_1 \underline{y}^1 + \mathbf{p}_0 \cdot (\omega_a^0 - \underline{y}^1) + \pi^1; \end{aligned}$$

$$(iv) \left(x_a^0; ((x_b^t, x_a^t), l^t, \omega_a^t)_{t \in \mathcal{T}}; (-y_l^t, -\underline{y}^t, \bar{y}^t)_{t \in \mathcal{T}} \right) \in F(E^\infty; \omega_a^0).$$

An intertemporal allocation rule φ^W is called *Walrasian* if and only if for each general OLG economy E^∞ associated with $\omega_a^0 \in \mathbb{R}_+^n$, and each

$$\left(x_a^0; ((x_b^t, x_a^t), l^t, \omega_a^t)_{t \in \mathcal{T}}; (-y_l^t, -\underline{y}^t, \bar{y}^t)_{t \in \mathcal{T}} \right) \in \varphi^W(E^\infty),$$

there exists a sequence of prices and wages $\{(\mathbf{p}_{t-1}, \mathbf{q}_t, w_t)\}_{t=1}^\infty$ such that the pair of the allocation and the sequence constitutes a Walrasian competitive equilibrium for E^∞ .

3 Two Axioms of *Hicksian optimism*

As discussed in Section 1 of this paper, moving from one Pareto-efficient allocation to another is likely to make someone worse off while making someone else better off, and the Pareto principle itself cannot be used for making welfare judgments in such a movement. *Hicksian optimism* is the neoclassical economic belief that consistent and sequential policy shifts that push society's utility possibility frontier upward in the 'north-east' direction, that is, consistent and sequential applications of "Pareto efficient policies" (Corden (1984)), will eventually make everyone better off.

3.1 Consistent and Sequential Applications of "Pareto Efficient Policies"

Inspired by such an idea, in what follows, we shall first introduce and define the 'improvements' that Hicks had in mind. After this discussion, we will formulate

in the next subsection, the *Hicksian optimism*, as axioms of (intertemporal) allocation rules.

For this purpose, we consider a general OLG economy, $E^\infty = (E_t = \langle Y^t; \omega_l; u \rangle)_{t=1}^\infty \in \mathcal{E}^\infty$, and a period economy $E^* = \langle Y^*; \omega_l; u \rangle \in \mathcal{E}$. We say that the general OLG economy $E^\infty = (E_t)_{t=0}^\infty$ associated with $\omega_a^0 \in \mathbb{R}_+^n$ is *growing and transforming* to a *stationary OLG economy* E^* if and only if

- (i) for all $t = 1, 2, \dots$, $\underline{Y}^t \subseteq \underline{Y}^{t+1}$ and $\omega_b^t \leq \omega_b^{t+1}$ hold, and
- (ii) $\lim_{t \rightarrow \infty} \underline{Y}^t = \underline{Y}^* \supset \underline{Y}^1$, $\lim_{t \rightarrow \infty} \omega_b^t = \omega_b^*$.

Here, the growing process from a period economy to the next period economy within a general OLG economy is characterized by condition (i), which represents the dynamic process of technical progress due to the success of technological innovation. This technological innovation can take place both in $\underline{Y}^t$ and Y_b^t , where $\omega_b^t \leq \omega_b^{t+1}$ represents a technological innovation in the latter production possibility set. Moreover, the condition (ii) requires that the growing economy converges to a period economy E^* . This implies that the growing OLG economy E^∞ transforms to a stationary OLG economy in the limit, where *the stationary OLG economy is characterized by the infinite repetition of a period economy* E^* .

In this formulation of the growing and converging transition, a sequence of each period's production possibility sets is given to meet the presumption of the *consistent application of "Pareto efficient policies"* defined by Corden (1984). This is because such a transition shifts the society's utility possibility frontier upward in a more 'north-eastern' direction. Setting aside the issue of how the R&D investment for technological innovation can be incentivized, as discussed in Section 1, we will focus on the issue whether and how such a dynamic transition of period-economies can improve individual and social welfare in the long run as captured by the idea of the Hicksian Optimism discussed in Section 1.

'The improvements' that Hicks had in mind are visualized as the improvements of the 'convergent stationary OLG economy' resulting from a sequence of growing period economies over the initial period economy. Our notion of a growing and transforming economy requires more than what Hicks had in mind: the expansion of production possibility sets occurs for any two periods, and is therefore much stronger than Hicks' requirement for the improvements of the economy.

3.2 What does it mean by "eventually everyone being better off"?

Before formally defining *Hicksian optimism*, we first explain what is meant to say that eventually everyone is better-off from the consistent and sequential application of Pareto-efficient policies over a long period of time means, as posited by *Hicksian optimism*. If we are talking about the consistent implementation of "Pareto-efficient policies" over a period spanning centuries, then the initial and eventual societies are likely composed of different individuals. Neverthe-

less, what does the statement, “eventually, everyone is better off”, mean in our context?

This becomes a distinct issue when defining *Hicksian optimism* within our economic model. In our OLG economic model, each individual survives only two periods: youth and old age. Therefore, while the (temporal) utility functions of the 0-th generation elderly and the 1-st generation youth in the initial economy are the same as those of the $(T-1)$ -th generation elderly and the T -th generation youth in the eventual stationary economy, they are entirely distinct individuals. What are the social welfare implications, if any, of comparing the welfare levels of the initial elderly (or youth) generation with those of the eventual elderly (or youth) generation?

Such welfare comparisons can be considered reasonable by interpreting Hicks’ original idea of optimism as concerning eventual Pareto improvements among agents representing every “position” in the society, rather than among all individuals as living beings. In our economic model, there are two positions: the elderly and the young in each period. Therefore, the eventual Pareto improvement in our model implies that the welfare of every position, both the elderly and the young, improves from the initial economy.

Incidentally, in OLG economic models, individual positions are typically defined and named as old age and youth. However, the meaning of these positions can be interpreted differently depending on the context of the economic resource allocation problem being examined. For example, in our OLG economic model, the youth position may be interpreted as the position of the working class, supplying labor to earn economic rewards. Similarly, the old age position may be interpreted as the ‘investor’, investing accumulated capital assets to earn profit income. This interpretation corresponds to examining an allocation rule φ primarily as a mechanism to address contemporaneous resource allocation problems in the capitalist economy.

Alternatively, when interpreting the allocation rule φ as a solution to inter-generational resource allocation problems, the old age position in each period’s economy can represent individuals who survive only in the current period. Conversely, the youth position primarily represents individuals who survive in both the current and next periods.

Corresponding to these different interpretations of the underlying resource allocation problems, different measures of welfare of individual positions could be applied to implement the aforementioned welfare comparison. In the context of contemporaneous resource allocation problems, for instance, it is reasonable to define this welfare comparison as a comparison of the welfare of the two positions in the initial and eventual economies using their respective period utility functions, v_a and v_b .

3.3 Formulation of Hicks’ original idea of optimism

Now, comparing the eventual stationary OLG economy with the initial period economy, we can formulate the principle requiring that none of the young and the old be made worse off:

Hicksian Optimism (HO): For any general OLG economy $E^\infty = \langle Y^t; \omega_l; u \rangle_{t=1}^\infty \in \mathcal{E}^\infty$ associated with $\omega_a^0 \in \mathbb{R}_+^n$ such that E^∞ is *growing and transforming* to a stationary OLG economy E^* , if $\left(x_a^0; ((x_b^t, x_a^t), l^t, \omega_a^t)_{t \in \mathcal{T}}; (-y_l^t, -\underline{y}^t, \bar{y}^t)_{t \in \mathcal{T}}\right) \in \varphi(E^\infty)$ has $\lim_{t \rightarrow \infty} (x_a^{t-1}; ((x_b^t, x_a^t), \omega_a^t); (-y_l^t, -\underline{y}^t, \bar{y}^t)) = (x_a^*; ((x_b^*, x_a^*), l^*, \omega_a^*); (-y_l^*, -\underline{y}^*, \bar{y}^*)) \in F(E^*; \omega_a^*)$, then

$$v_b(x_b^*) \geq v_b(x_b^1) \text{ and } v_a(x_a^*) \geq v_a(x_a^0).$$

According to the axiom **Hicksian Optimism (HO)**, the improvements based on ‘growth policies’ should leave no one behind—all young and old generations will be made at least as well-off as their ancestors who existed before implementing such policies.⁷

Note that it is not required that everyone should (weakly) benefit at each period from the growth policies. It merely requires that, eventually and this could be a long time requiring “a great deal of human patience, more patience than is characteristic of the twentieth century, even of the economists of the twentieth century; more patience, perhaps, than we ought to ask” as observed by Hicks (1941, p.111) himself, no one is going to get hurt. Since our notion of a growing and transforming economy demands more than what Hicks requires for the growing of the economy, our formulations of the Hicksian optimism is therefore much weaker than what Hicks had in mind, making our versions of the Hicksian optimism even more appealing.

Note also that the axiom **(HO)** defines the welfare comparison of the elderly and the youth in initial economy with those in the eventual economy by means of their respective period utility functions, v_a and v_b . This implies that the axiom **(HO)** can be appropriately applied to the context of contemporaneous resource allocation problems. This formulation remains faithful to Hicks’s original idea of optimism, which was that the welfare of every individual position achieved under contemporaneous resource allocation problems could be improved by implementing “Pareto efficient policies” iteratively over the long term.

3.4 Formulation of Hicksian optimism in the context of intergenerational allocation problems

We now consider how Hicks’ original idea of optimism can be applied to intergenerational allocation problems. In this context, we compare the initial welfare allocation of the older generation (generation $t = 0$) and the younger generation (generation $t = 1$) with the welfare allocation of both generations in the eventual stationary economy. In the context of intergenerational allocation, note that the

⁷It may be of interest to note that our idea that ‘no one is left behind’ exposed in the Hicksian Optimism is similar to that advocated and used by the UNDP: “People get left behind when they lack the choices and opportunities to participate in and benefit from development progress.” (UNDP, 2018)

older generation tends to be short-sighted, while the younger generation tends to be more farsighted. Given this, it is reasonable to define the welfare of the older individual in each period using the utility function v_a . In contrast, the welfare of the younger individual can be defined in two ways: using the period utility function v_b or the individual's lifetime utility function u . Axiom **HO** in the last subsection uses v_b to formalize the *Hicksian optimism*. On the other hand, using the lifetime utility function u can be justified by the farsightedness of the younger generation. As mentioned in Section 1, it is reasonable to argue that the welfare of the younger generation should be measured by her lifetime utility rather than her utility at the youth, since she will live in both her initial and subsequent periods. This allows her to compensate for her disadvantageous welfare in her youth with her advantageous welfare in her old age.

Thus, we may have an alternative formulation of *Hicksian optimism* as follows:

Hicksian Optimism with Lifetime Welfare (HOLW): For any general OLG economy $E^\infty = \langle Y^t; \omega_l; u \rangle_{t=1}^\infty \in \mathcal{E}^\infty$ associated with $\omega_a^0 \in \mathbb{R}_+^n$ such that E^∞ is *growing and transforming* to a stationary OLG economy E^* , if $(x_a^0; ((x_b^t, x_a^t), l^t, \omega_a^t)_{t \in \mathcal{T}}; (-y_l^t, -\underline{y}^t, \bar{y}^t)_{t \in \mathcal{T}}) \in \varphi(E^\infty)$ has

$$\lim_{t \rightarrow \infty} (x_a^{t-1}; ((x_b^t, x_a^t), \omega_a^t); (-y_l^t, -\underline{y}^t, \bar{y}^t)) = (x_a^*; ((x_b^*, x_a^*), l^*, \omega_a^*); (-y_l^*, -\underline{y}^*, \bar{y}^*)) \in F(E^*; \omega_a^*),$$

then

$$v_a(x_a^*) \geq v_a(x_a^0) \text{ and } u(x_b^*, x_a^*) = U(v_b(x_b^*), v_a(x_a^*)) \geq U(v_b(x_b^1), v_a(x_a^1)) = u(x_b^1, x_a^1).$$

Note that **HO** and **HOLW** are formally independent since, (1) given $v_a(x_a^*) \geq v_a(x_a^0)$, $v_b(x_b^*) \geq v_b(x_b^1)$ does not necessarily imply $u(x_b^*, x_a^*) \geq u(x_b^1, x_a^1)$, and (2) given $v_a(x_a^*) \geq v_a(x_a^0)$, $u(x_b^*, x_a^*) \geq u(x_b^1, x_a^1)$ does not necessarily imply $v_b(x_b^*) \geq v_b(x_b^1)$ either.

Our axioms of **HO** and **HOLW** formally may be viewed as similar to various monotonicity properties proposed in the related literatures of axiomatic bargaining (Kalai (1977)) and of resource allocations (see, for example, Chambers and Hayashi (2020), Chun and Thompson (1988), Moulin and Roemer (1989), Moulin and Thompson (1988), and Roemer (1986)). We note, however, that there are several substantial differences between those monotonicity properties and our notions of *Hicksian optimism*. The monotonicity properties introduced in those literatures deal with two different static economies and thus require monotonicity across different economic environments, and the underlying allocation rules are static as well. In our case, the axioms of **HO** and **HOLW** are applied to each simple OLG economy so that it operates within a given economic environment, and the allocation rules are intertemporal. These differences will make our main impossibility results (see the next section) intrinsically different from the impossibility results obtained in the above literature on resource allocations, from both conceptual and mathematical viewpoints.

4 Any Efficient and Individually Rational Allocation Rule Is Not Hicksian Optimistic

In this section, we examine the fate of the *Hicksian optimism* in the context of certain mechanisms of allocating resources. For this purpose, we introduce the following two versions of axioms for efficiency of intertemporal allocation rules that are to be embedded in the class of resource allocation mechanisms that we study.

Pareto Efficiency (PE): For any general OLG economy $E^\infty = \langle Y^t; \omega_l; u \rangle_{t=1}^\infty \in \mathcal{E}^\infty$ associated with ω_a^0 , and every intertemporal allocation

$\left(x_a^0; ((x_b^t, x_a^t), l^t, \omega_a^t)_{t \in \mathcal{T}}; (-y_l^t, -\underline{y}^t, \bar{y}^t)_{t \in \mathcal{T}} \right)$ in $\varphi(E^\infty)$, there is no other $\left(x_a'^0; ((x_b'^t, x_a'^t), l'^t, \omega_a'^t)_{t \in \mathcal{T}}; (-y_l'^t, -\underline{y}'^t, \bar{y}'^t)_{t \in \mathcal{T}} \right)$ in $F(E^\infty; \omega_a^0)$ such that:

$v_a(x_a'^0) \geq v_a(x_a^0)$, $u(x_b'^t, x_a'^t) \geq u(x_b^t, x_a^t)$ for all $t \in \mathcal{T}$ and either $v_a(x_a'^0) > v_a(x_a^0)$ or $u(x_b'^t, x_a'^t) > u(x_b^t, x_a^t)$ for some $t \in \mathcal{T}$.

Weak Pareto Efficiency (WPE): For any general OLG economy $E^\infty = \langle Y^t; \omega_l; u \rangle_{t=1}^\infty \in \mathcal{E}^\infty$ associated with ω_a^0 , and every intertemporal allocation

$\left(x_a^0; ((x_b^t, x_a^t), l^t, \omega_a^t)_{t \in \mathcal{T}}; (-y_l^t, -\underline{y}^t, \bar{y}^t)_{t \in \mathcal{T}} \right)$ in $\varphi(E^\infty)$, there is no other $\left(x_a'^0; ((x_b'^t, x_a'^t), l'^t, \omega_a'^t)_{t \in \mathcal{T}}; (-y_l'^t, -\underline{y}'^t, \bar{y}'^t)_{t \in \mathcal{T}} \right)$ in $F(E^\infty; \omega_a^0)$ such that:

- (i) $v_a(x_a'^0) \geq v_a(x_a^0)$, $u(x_b'^t, x_a'^t) \geq u(x_b^t, x_a^t)$ for all $t \in \mathcal{T}$ and either $v_a(x_a'^0) > v_a(x_a^0)$ or $u(x_b'^t, x_a'^t) > u(x_b^t, x_a^t)$ for some $t \in \mathcal{T}$, and
- (ii) $(x_b'^t, x_a'^t) = (x_b^t, x_a^t)$ except for a finite number of t .

Moreover, we also introduce an axiom for participation incentives. Consider an autarkic economy in period t . In this economy the generation t at the young age produces ω_b^t by using one unit of the labor endowment. Then, they allocate it optimally for consumption in their youth and old ages:

$$\max_{z_b, z_a} u(z_b, z_a)$$

subject to:

$$z_b + z_a = \omega_b^t; \quad z_b, z_a \in \mathbb{R}_+^n.$$

Let $(z_b^t, z_a^t) \in \mathbb{R}_+^n \times \mathbb{R}_+^n$ be the solution to the above optimization problem that the generation t could choose to consume if they lived in the autarkic economy. Then, the axiom for participation incentives is formulated as follows:

Individual Rationality (IR): For any general OLG economy $E^\infty = \langle Y^t; \omega_l; u \rangle_{t \in \mathcal{T}}$ in $\mathcal{E}^\infty$ associated with ω_a^0 , and every $\left(x_a^0; ((x_b^t, x_a^t), l^t, \omega_a^t)_{t \in \mathcal{T}}; (-y_l^t, -\underline{y}^t, \bar{y}^t)_{t \in \mathcal{T}} \right)$ in $\varphi(E^\infty)$, the following conditions hold:

$$\begin{aligned} u(x_b^t, x_a^t) &\geq u(z_b^t, z_a^t) \text{ for every generation } t \in \mathcal{T}, \\ \text{and } v_a(x_a^0) &\geq v_a(\omega_a^0) \text{ for generation } t = 0. \end{aligned}$$

Weak Pareto Efficiency (**WPE**) is introduced by Balasko and Shell (1980) in the context of OLG models. It is weaker than the standard Pareto efficiency condition (**PE**): in (**WPE**), the Pareto domination test is applied only to certain types of feasible intertemporal allocations in which the consumption bundles are identical except for a finite number of generations. Following the works of Balasko and Shell (1980, Proposition 4.4) and of Tvede (2010, Corollary 7.9 and Corollary 7.10),⁸ it can be shown that, in our OLG model, a Walrasian competitive equilibrium allocation satisfies **WPE**, and any feasible intertemporal allocation satisfying **WPE** can be a Walrasian competitive equilibrium allocation. It may be noted that, since our result is an impossibility involving **WPE** (see Theorems 1 and 3 below), it is sufficient to formulate this weaker version of the Pareto efficiency condition.

Individual Rationality (**IR**) requires that an intertemporal allocation rule be such that, for every period $t \in \mathcal{T}$, every agent in the population of this period has an incentive to participate in social economic activities of the period economy E_t : by participating in social economic activities of the period economy, the agent would be at least as well-off as by choosing to engage in her own autarkic economy. Note that in every period $t \in \mathcal{T}$, generation $t = 0$ at the old age can choose an autarkic economic action by consuming her savings vector ω_a^0 , while generation t at the young age can produce ω_b^t by operating Y_b^t with her one unit labor supply and consuming ω_b^t in her autarkic economy.⁹

4.1 Incompatibility of WPE and IR with HO

The two axioms, **WPE** and **IR**, capture a broader class of intertemporal allocation rules. Can any such intertemporal allocation rules deliver the *Hicksian optimism*? The following result provides an answer to this question and the proof can be found in the Appendix.

Theorem 1 *Suppose $\omega_a^0 > \mathbf{0}$ and $\omega_b^1 > \mathbf{0}$. Then, there exists no intertemporal allocation rule satisfying **Hicksian Optimism**, **Weak Pareto Efficiency**, and **Individual Rationality** simultaneously.*

Therefore, Theorem 1 suggests that any resource allocation mechanism characterized by an efficient and individually rational intertemporal allocation rule is bound to leave some individual behind even if the economy is growing and expanding due to technological innovation and even after a long period of time.

⁸While Balasko and Shell (1980) focus on pure exchange OLG model, Tvede (2010) also discusses OLG economies with production.

⁹Note that we could have introduced a stronger individual rationality condition than this version of **IR** by incorporating information concerning individuals' ownerships of production technologies in $\underline{Y}^t$. Since our results are impossibility results (see Theorems 1 and 3 below), it becomes unnecessary to formulate a stronger individual rationality condition.

As shown in Balasko and Shell (1980, Proposition 4.4) and Tvede (2010, Corollary 7.9), in OLG models, the Walrasian intertemporal allocation rule is weakly Pareto efficient. It is clear that the Walrasian intertemporal allocation rule is also individually rational. Then, by Theorem 1, we obtain the following result.

Corollary 2 *Suppose $\omega_a^0 > \mathbf{0}$ and $\omega_b^1 > \mathbf{0}$. Then, the Walrasian intertemporal allocation rule does not satisfy **Hicksian Optimism**.*

Thus, the market mechanism does not deliver the *Hicksian optimism*: in a market system, economic expansions can leave some individuals behind—some individuals would be made worse off than his ancestor in the pre-expansion economy.

4.2 Incompatibility of WPE and IR with HOLW

The above impossibility results are primarily relevant in the context of contemporaneous allocation problems. In this subsection, we examine whether an efficient and individually rational allocation rule can deliver *Hicksian optimism* when applied to intergenerational resource allocation problems. The following result provides an answer to this examination and the proof can be again found in the Appendix.

Theorem 3 *Suppose $\omega_a^0 > \mathbf{0}$ and $\omega_b^1 > \mathbf{0}$. Then, there exists no intertemporal allocation rule satisfying **Hicksian Optimism with Lifetime Welfare, Weak Pareto Efficiency, and Individual Rationality** simultaneously.*

Thus, Theorem 3 suggests that even in the context of intergenerational allocation problems, any efficient and individually rational allocation mechanism fails to deliver the *Hicksian optimism*.

Again, as a corollary of the above theorem, we can confirm that the market mechanism does not deliver the *Hicksian optimism* in the context of intergenerational allocation problems.

Corollary 4 *Suppose $\omega_a^0 > \mathbf{0}$ and $\omega_b^1 > \mathbf{0}$. Then, the Walrasian intertemporal allocation rule does not satisfy **Hicksian Optimism with Lifetime Welfare**.*

5 Conclusion

In a simple OLG economy, we have shown that any resource allocation mechanism characterized by a weakly efficient and individually rational intertemporal allocation rule faces a dilemma: with continued technological innovation and progress, some type of individuals is bound to be left behind even after a long period of time. Since the market mechanism characterized by the Walrasian allocation rule belongs to the class of weakly efficient and individually rational intertemporal allocation rules, the market mechanism faces the same dilemma: it cannot deliver the Hicksian optimism—an ideal as envisioned by Hicks (1941) and, more recently, as advocated by the UNDP (2015) in its 2030 Agenda for Sustainable Development.

Pareto efficiency is deeply rooted in economics and is highly appealing for resource allocation. Given the tremendous technological innovation made in recent human history and the great economic progress made in recent decades, the Hicksian optimism seems a very attractive property to be demanded for resource allocation mechanisms. If we follow these lines of reasoning, the property of individual rationality needs a careful and possibly critical examination.

To begin with, we note that, as our results in Section 4 show, the unfettered market mechanism characterized by the Walrasian allocation rule cannot deliver Hicksian optimism. Part of the reasons is that the Walrasian intertemporal allocation rule is both weakly efficient and individually rational. Suppose we abandon individual rationality but still want to use the market mechanism to allocate resources. One possibility is to use the market mechanism coupled with a redistributive scheme (perhaps similar to that required for the second welfare theorem) to ensure that, eventually, every generation, being young or old, is at least as well-off as his/her ancestor who existed at the very beginning.

Resource-wise, this is feasible since the economy is expanding and everyone's eventual initial endowment is bigger than the initial endowment of his/her ancestor in period 0 or period 1. Of course, the Walrasian intertemporal allocation rule coupled with a redistributive scheme may violate individual rationality introduced in Section 4, the reason being that, some individuals may be made worse off than their initial endowments in that period after redistributions.

If, on the other hand, one wants to go beyond the market mechanism, there are several studies on certain egalitarian type allocation rules for specific static economies involving one input and one output, where it has been shown that the proposed allocation rules are Pareto efficient and satisfy some technological monotonicities. See the contributions by Moulin (1987, 1990), Moulin and Roemer (1989), and Roemer and Silvestre (1987)¹⁰. An important reason why positive results can be obtained in those settings is that their static economies

¹⁰Note, however, that these results are quite fragile. As Chambers and Hayashi (2020, Proposition 1) show, even in economies with two goods where both goods are commodities that can switch roles as inputs and outputs according to each economy, no rule satisfies Pareto efficiency, technological monotonicity, and the free access lower bound. Furthermore, in economies with two inputs and one output, the same impossibility result holds (see Chambers and Hayashi (2020, Proposition 2)).

involve one input (which is a nontransferable productive factor, like labor) and one output and, for such specific economies, the property of individual rationality almost has no bite. For our economy, individual rationality is non-trivial and is rather stringent.

6 References

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Appendix

6.1 Proof of Theorem 1

Proof. Suppose that there are only two commodities. Given $\varepsilon \in (0, \frac{1}{2})$, let $\alpha \in \mathbb{R}_+$ be such that $\alpha > \frac{1-\varepsilon}{\varepsilon}$, which implies that $\alpha > 1$. Let δ be sufficiently small with $0 < \delta < \alpha - 1$. Let $u = U(v_b, v_a)$ satisfy the following properties:¹¹ for any $(x_1, x_2) \in \mathbb{R}_+^2$,

$$v_b(x_1, x_2) \equiv \min \left\{ \frac{x_1}{\alpha} (1 + \delta), x_2 \right\} \text{ and } v_a(x_1, x_2) \equiv \min \left\{ x_1, \frac{x_2}{\alpha} (1 + \delta) \right\},$$

and U is increasing in $\mathbb{R}^2$ such that it meets the following indifference relations (see Figure 1 for an illustration of the indifference curves of the constructed U function): (i) consider two points, $(\varepsilon(1 + \delta), \frac{\varepsilon}{\alpha}(1 + \delta)^2)$ and $(\frac{\varepsilon}{\alpha}(1 + \delta)^2, \varepsilon(1 + \delta))$,

¹¹The specific one-period utility functions are used here to provide a simplest economy to establish our result. We may note that, at the expense of increasing complexity, the proof goes through with a more standard type of one-period utility functions, where the utility functions are strongly monotonic and are not the Leontief preferences of the two goods.

in the (v_b, v_a) -space, for any (v'_b, v'_a) located at the convex hull of $\{(\varepsilon(1+\delta), \frac{\varepsilon}{\alpha}(1+\delta)^2), (\frac{\varepsilon}{\alpha}(1+\delta)^2, \varepsilon(1+\delta))\}$,

$$\begin{aligned} U\left(\varepsilon(1+\delta), \frac{\varepsilon}{\alpha}(1+\delta)^2\right) &= U\left(\frac{\varepsilon}{\alpha}(1+\delta)^2, \varepsilon(1+\delta)\right) \\ &= U(v'_b, v'_a) \\ &= U\left(\frac{\varepsilon}{\alpha}(1+\delta)^2, v''_a\right) \text{ (for any } v''_a \geq \varepsilon(1+\delta)), \\ &= U\left(v''_b, \frac{\varepsilon}{\alpha}(1+\delta)^2\right) \text{ (for any } v''_b \geq \varepsilon(1+\delta)); \end{aligned}$$

(ii) for any $\pi > 0$ with $\frac{\varepsilon}{\alpha}(1+\delta)^2 < \pi < \varepsilon(1+\delta)$, and for any (v'_b, v'_a) located at the convex hull of $\{(\varepsilon(1+\delta), \pi), (\pi, \varepsilon(1+\delta))\}$

$$\begin{aligned} U(\varepsilon(1+\delta), \pi) &= U(\pi, \varepsilon(1+\delta)) \\ &= U(v'_b, v'_a) \\ &= U(\pi, v''_a) \text{ (for any } v''_a \geq \varepsilon(1+\delta)), \\ &= U(v''_b, \pi) \text{ (for any } v''_b \geq \varepsilon(1+\delta)); \end{aligned}$$

and (iii) for any $(v'_b, v'_a) \geq (\varepsilon(1+\delta), \varepsilon(1+\delta))$, $U(v'_b, v'_a) = \min\{v'_b, v'_a\}$.

Insert Figure 1.

Let a production possibility set Y^t at $t \in \mathcal{T}$ be represented by a von Neumann production technology (A^t, B^t, L^t) , where A^t is a matrix of material input coefficients; B^t is a matrix of gross output coefficients; and L^t is a vector of direct labor input coefficients. In particular, let $\{(A^t, B^t, L^t)\}_{t=1, \dots}$ be specified as follows:

$$\begin{aligned} A^1 &\equiv \begin{bmatrix} 2\varepsilon & 0 \\ 2\varepsilon & 0 \end{bmatrix}, \quad B^1 \equiv \begin{bmatrix} G_1(t) + 2\varepsilon & 2\varepsilon \\ G_2(t) + 2\varepsilon & 2\varepsilon \end{bmatrix}, \quad L^1 \equiv (1, 1), \text{ and} \\ A^t &\equiv \begin{bmatrix} \omega_{b1}^t & 0 \\ \omega_{b2}^t & 0 \end{bmatrix}, \quad B^t \equiv \begin{bmatrix} G_1(t) + \omega_{b1}^{t+1} & \omega_{b1}^t \\ G_2(t) + \omega_{b2}^{t+1} & \omega_{b2}^t \end{bmatrix}, \quad L^t \equiv (1, 1) \quad (\forall t = 2, \dots), \end{aligned}$$

and

$$\begin{aligned} \omega_b^t &= (\omega_{b1}^t, \omega_{b2}^t) = 2\varepsilon \cdot \left[\alpha^{\frac{t-2}{t+2\alpha}}, (1+\delta)^{\frac{t-2}{t+2\alpha}} \right] \text{ for every } t = 2, \dots, \text{ and} \\ \omega_a^0 &= (\omega_{a1}^0, \omega_{a2}^0) = 2\varepsilon(1, 1), \end{aligned}$$

where

$$(G_1(t), G_2(t)) = \begin{cases} (1, 1) & \text{if } t = 1 \\ \left(\alpha^{\frac{t-2}{t+2\alpha}}, (1+\delta)^{\frac{t-2}{t+2\alpha}} \right) & \text{if } t = 2, \dots \end{cases}$$

By this construction, it is obvious that $G_1(t)$ is increasing at every $t = 1, \dots$, such that $G_1(2) = 1$ and $\lim_{t \rightarrow \infty} G_1(t) = \alpha$; and $G_2(t)$ is increasing at every $t = 1, \dots$, such that $G_2(2) = 1$ and $\lim_{t \rightarrow \infty} G_2(t) = 1 + \delta$; and moreover:

$$(G_1(t), G_2(t)) > \omega_b^t \text{ for every } t = 1, \dots$$

Note that $\lim_{t \rightarrow \infty} (A^t, B^t, L^t) = (A^*, B^*, L^*)$ holds, where

$$A^* = \begin{bmatrix} \omega_{b1}^* & 0 \\ \omega_{b2}^* & 0 \end{bmatrix}, \quad B^* = \begin{bmatrix} \alpha + \omega_{b1}^* & \omega_{b1}^* \\ (1 + \delta) + \omega_{b2}^* & \omega_{b2}^* \end{bmatrix}, \quad L^* = (1, 1),$$

in which $\omega_b^* = (\omega_{b1}^*, \omega_{b2}^*) = 2\varepsilon(\alpha, 1 + \delta)$ holds. Also, note that $A^2 = A^1$, as $\omega_{b1}^2 = 2\varepsilon = \omega_{b2}^2$ holds by construction. For each (A^t, B^t, L^t) , let

$$Y_{(A^t, B^t, L^t)} \equiv \{y = (-y_l, -y, \bar{y}) \in \mathbb{R}_- \times \mathbb{R}_-^2 \times \mathbb{R}_+^2 \mid \exists z \in \mathbb{R}_+^2 : y \leq (-L^t z, -A^t z, B^t z)\},$$

and set $Y^t \equiv Y_{(A^t, B^t, L^t)}$. It can be checked that Y^t satisfies **A1**, **A2**, **A3**. In particular, Y_b^t can be derived from the 2nd column production process of (A^t, B^t, L^t) while Y^t can be derived from the 1st column production process of (A^t, B^t, L^t) . Moreover, for any $t = 1, \dots$, $Y^t \subseteq Y^{t+1}$ and $\omega_b^t \leq \omega_b^{t+1}$ hold.

Let an intertemporal allocation rule φ satisfy **WPE**, **IR**, and **HO**. Note that the above defined general OLG economy $E^\infty = (E_t = \langle Y^t; \omega_l; u \rangle)_{t=1}^\infty \in \mathcal{E}^\infty$ associated with $\omega_a^0 = (2\varepsilon, 2\varepsilon)$ is growing and transforming to the stationary OLG economy $E^* = \langle Y^*; \omega_l; u \rangle$, where $Y^* \equiv Y_{(A^*, B^*, L^*)}$. Note that, as $2\varepsilon < 1$ by $\varepsilon \in (0, \frac{1}{2})$, $2\varepsilon(\alpha, 1 + \delta) < (\alpha, 1 + \delta)$ holds, which implies that Y^* satisfies **A1(ii)**.

Let $\mathbf{e}_1 \equiv \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Let $(x_a^0; (x^t, l^t, \omega_a^t)_{t=1, \dots}; (y^t)_{t=1, \dots}) \in \varphi(E^\infty)$ be such that, for each period $t \in \mathcal{T}$,

$$l^t = 1; \quad \omega_a^t = \omega_b^{t+1}; \quad y^t = (-L^t \mathbf{e}_1, -A^t \mathbf{e}_1, B^t \mathbf{e}_1); \quad \text{and } x_a^{t-1} + x_b^t + \omega_a^t = (B^t - A^t) \mathbf{e}_1 + \omega_a^{t-1},$$

and $\lim_{t \rightarrow \infty} (x_a^{t-1}; (x^t, l^t, \omega_a^t); y^t) = (x_a^*; (x^*, l^*, \omega_a^*); y^*) \in F(E^*; \omega_a^*)$, where $\omega_a^* = \omega_b^*$.

Given $\omega_b^* = 2\varepsilon(\alpha, 1 + \delta)$, the utility possibility set of autarkic economic activities with Y^* is defined as follows:

$$\mathcal{V}(\omega_b^*) \equiv \{(v_b(z^b), v_a(z^a)) \in \mathbb{R}_+^2 \mid \exists (z^b, z^a) \in \mathbb{R}_+^2 \times \mathbb{R}_+^2 : z^b + z^a \leq \omega_b^*\}.$$

Then, by the definitions of one-period utility functions $v_b(\cdot), v_a(\cdot)$ and the fact that $\alpha > 1 - \delta$ (so that $2\varepsilon\alpha > 2\varepsilon(1 + \delta)$ holds), $\mathcal{V}(\omega_b^*)$ is characterized as follows:

$$\mathcal{V}(\omega_b^*) = \left\{ (v'_b, v'_a) \in \mathbb{R}_+^2 \mid -\frac{1 + \delta}{\alpha} v'_b + 2\varepsilon \frac{(1 + \delta)^2}{\alpha} \leq v'_a \right\}.$$

Then, by the definition of U ,

$$\left(\varepsilon(1 + \delta), \frac{\varepsilon}{\alpha}(1 + \delta)^2 \right) = \arg \max_{(v'_b, v'_a) \in \mathcal{V}(\omega_b^*)} U(v'_b, v'_a),$$

and the optimal autarkic allocation (z^{*b}, z^{*a}) at the stationary OLG economy E^* is $(z^{*b}, z^{*a}) \equiv (\varepsilon(a, 1 + \delta), \varepsilon(a, 1 + \delta))$, which achieves $(v_b(z^{*b}), v_a(z^{*a})) = (\varepsilon(1 + \delta), \frac{\varepsilon}{\alpha}(1 + \delta)^2)$.

Note that at the stationary OLG economy E^* , a stationary allocation $(x_a; (x, l^*, \omega_a^*); y^*) \in F(E^*; \omega_a^*)$ can be efficient and individually rational if and only if $x = (x_b, x_a)$ is a solution to the following optimization problem:

$$\max U(v_b(x'_b), v_a(x'_a)) \text{ such that } x'_b + x'_a \leq (\alpha, 1 + \delta).$$

In this case, the corresponding utility possibility set is:

$$\begin{aligned} \mathcal{V}(\alpha, 1 + \delta) &\equiv \{(v_b(x'_b), v_a(x'_a)) \in \mathbb{R}_+^2 \mid x'_b + x'_a \leq (\alpha, 1 + \delta)\} \\ &= \left\{ (v'_b, v'_a) \in \mathbb{R}_+^2 \mid -\frac{1 + \delta}{\alpha} v'_b + \frac{(1 + \delta)^2}{\alpha} \leq v'_a \right\}. \end{aligned}$$

Thus, the above optimization problem is reduced to:

$$\max U(v'_b, v'_a) \text{ such that } (v'_b, v'_a) \in \mathcal{V}(\alpha, 1 + \delta).$$

Then, by the definition of the indifference curve of $u = U(v'_b, v'_a)$, it follows that:

$$\left(\varepsilon(1 + \delta), \frac{(1 - \varepsilon)(1 + \delta)^2}{\alpha} \right) = \arg \max_{(v'_b, v'_a) \in \mathcal{V}(\alpha, 1 + \delta)} U(v'_b, v'_a).$$

By **WPE** and **IR**, $(v_b(x_b^*), v_a(x_a^*)) = \left(\varepsilon(1 + \delta), \frac{(1 - \varepsilon)(1 + \delta)^2}{\alpha} \right)$ must hold¹².

Then, by **HO**, we thus obtain

$$(\mathbf{B.1}) \quad v_a(x_a^*) = \frac{(1 - \varepsilon)(1 + \delta)^2}{\alpha} \geq v_a(x_a^0) \text{ and } v_b(x_b^*) = \varepsilon(1 + \delta) \geq v_b(x_b^1).$$

Next, since $(G_1(1), G_2(1)) = (G_1(2), G_2(2))$ and $\omega_b^2 = (2\varepsilon, 2\varepsilon)$,

$$\begin{aligned} &\mathcal{V}(G_1(1), G_2(1)) \\ &= \mathcal{V}(G_1(2), G_2(2)) = \{(v_b(x'_b), v_a(x'_a)) \in \mathbb{R}_+^2 \mid x'_b + x'_a \leq (1, 1)\} \\ &= \text{convex hull of } \left\{ (0, 0), \left(\frac{(1 + \delta)}{\alpha}, 0 \right), \left(\frac{1 + \delta}{1 + \alpha + \delta}, \frac{1 + \delta}{1 + \alpha + \delta} \right), \left(0, \frac{(1 + \delta)}{\alpha} \right) \right\}. \end{aligned}$$

This is the set of feasible utility allocations between the young age and the old age, given that $\omega_a^0 = \omega_b^1 = (2\varepsilon, 2\varepsilon)$ is productively invested in the production in period 1 economy E^1 and ω_b^2 is allowed to be inherited and productively invested in period 2 economy E^2 , and ω_b^3 is allowed to be inherited for the producing in period 3 economy E^3 .

By **IR** of generation $t = 0$, $v_a(x_a^0) \geq v_a(\omega_a^0) = \frac{2\varepsilon(1 + \delta)}{\alpha}$ must hold, which implies that $v_a(x_a^0) \geq v_a\left(\frac{2\varepsilon(1 + \delta)}{\alpha}, 2\varepsilon\right)$. The last result in turn implies that $v_b(x_b^1) \leq v_b\left(1 - \frac{2\varepsilon(1 + \delta)}{\alpha}, 1 - 2\varepsilon\right)$ must hold. By definition of v_b , $v_b\left(1 - \frac{2\varepsilon(1 + \delta)}{\alpha}, 1 - 2\varepsilon\right) \leq 1 - 2\varepsilon$ holds.

¹²This property follows from the fact that the efficient and individually rational correspondence, that is, the allocation rule selecting all efficient and individually rational allocations at each and every economy, is upper hemi-continuous.

Let $\varepsilon \in (\frac{1}{3}, \frac{1}{2})$. Then, $\frac{1-\varepsilon}{1-2\varepsilon} > \frac{1-\varepsilon}{\varepsilon}$ since $1-2\varepsilon < \varepsilon$ if $\varepsilon \in (\frac{1}{3}, \frac{1}{2})$. Choose $\alpha \in (\frac{1-\varepsilon}{\varepsilon}, \frac{1-\varepsilon}{1-2\varepsilon}]$. Then, $1-2\varepsilon \leq \frac{1-\varepsilon}{\alpha}$ and $1-2\varepsilon \leq \frac{(1-\varepsilon)(1+\delta)^2}{\alpha}$. Thus, $v_b(x_b^1) \leq \frac{(1-\varepsilon)(1+\delta)^2}{\alpha}$ holds. Thus, we have:

$$(B.2) \quad (v_b(x_b^1), v_a(x_a^0)) \leq \left(\frac{(1-\varepsilon)(1+\delta)^2}{\alpha}, \frac{(1-\varepsilon)(1+\delta)^2}{\alpha} \right).$$

To complete our proof, we now construct an intertemporal feasible allocation $(x_a^0; (x^t, l^t, \omega_a^t)_{t=1, \dots}; (y^t)_{t=1, \dots}) \in F(E^\infty; \omega_a^0)$ such that

$$\begin{aligned} (x_a^0; (x_b^1, x_a^1), l^1, \omega_a^1; y^1) &= (x_a^0; (x_b^1, x_a^1), 1, \omega_a^1; y^1) \\ \text{with } (x_b^1, x_a^0) &= \left(\left(\frac{\alpha}{\alpha+1}, \frac{1}{\alpha+1} \right), \left(\frac{1}{\alpha+1}, \frac{\alpha}{\alpha+1} \right) \right) \text{ at } E_1; \\ \text{and } (x_a^{t-1}; (x_b^t, x_a^t), l^t, \omega_a^t; y^t) &= (x_a^{t-1}; (x_b^t, x_a^t), 1, \omega_a^t; y^t) \text{ at } E_t \quad (\forall t = 2, \dots). \end{aligned}$$

Note that $(x_a^0; (x_b^1, x_a^1), l^1, \omega_a^1; y^1)$ is a temporary feasible for E_1 with ω_a^0 , since $x_a^0 + x_b^1 + \omega_a^1 = (1, 1) + \omega_a^1 = (1, 1) + (2\varepsilon, 2\varepsilon) = (B^1 - A^1) \mathbf{e}_1 + \omega_a^0$ holds, where $\omega_a^1 = (2\varepsilon, 2\varepsilon)$ comes from $\omega_a^1 = \omega_a^2$ by definition of $(x_a^0; (x^t, l^t, \omega_a^t)_{t=1, \dots}; (y^t)_{t=1, \dots})$ and from $\omega_b^2 = (2\varepsilon, 2\varepsilon)$ by construction of B^t . Thus, the intertemporal allocation $(x_a^0; (x^t, l^t, \omega_a^t)_{t=1, \dots}; (y^t)_{t=1, \dots})$ is indeed feasible for E^∞ with ω_a^0 , since it is identical to the φ -optimal allocation $(x_a^0; (x^t, l^t, \omega_a^t)_{t=1, \dots}; (y^t)_{t=1, \dots})$ except for the period $t = 1$.

Then, $v_b(x_b^1) = v_a(x_a^0) = \min \left\{ \frac{\alpha}{\alpha+1} \frac{(1+\delta)}{\alpha}, \frac{1}{\alpha+1} \right\} = \frac{1}{\alpha+1}$. However,

$$\frac{1}{\alpha+1} - \frac{(1+\delta)^2(1-\varepsilon)}{\alpha} > 0 \text{ for the sufficiently small } \delta > 0,$$

as $\frac{1}{\alpha+1} - \frac{(1-\varepsilon)}{\alpha} = \frac{\alpha\varepsilon + \varepsilon - 1}{\alpha(\alpha+1)} > 0$ by $\alpha > \frac{1-\varepsilon}{\varepsilon}$. By (B.1) and (B.2), we must have

$$(B.3) \quad v_a(x_a^0) > v_a(x_a^1) \text{ and } v_b(x_b^1) > v_b(x_b^1).$$

Since $u(\cdot, \cdot)$ is increasing, we must also have

$$(B.4) \quad u(x_b^1, x_a^1) > u(x_b^1, x_a^1).$$

Then, the φ -optimal allocation $(x_a^0; (x^t, l^t, \omega_a^t)_{t=1, \dots}; (y^t)_{t=1, \dots})$ cannot be weakly Pareto efficient, a contradiction. ■

6.2 Proof of Theorem 3

Proof. The proof of Theorem 3 is similar to that of Theorem 1 and we outline the proof strategy briefly below. First, construct the exactly same economies $E^\infty = (E_t = \langle Y^t; \omega_l; u \rangle)_{t=1}^\infty \in \mathcal{E}^\infty$ associated with $\omega_a^0 = (2\varepsilon, 2\varepsilon)$ and $E^* =$

$\langle Y^*; \omega_l; u \rangle$ associated with $\omega_a^* = 2\varepsilon(\alpha, 1 + \delta)$ as those in the proof of Theorem 1. Let an intertemporal allocation rule φ satisfy **WPE**, **IR**, and **HOLW**. Then, by the same argument as developed in the proof of Theorem 1, it follows from **WPE** and **IR** that $(v_b(x_b^*), v_a(x_a^*)) = \left(\varepsilon(1 + \delta), \frac{(1-\varepsilon)(1+\delta)^2}{\alpha}\right)$ must hold.

Then, by **HOLW**, we thus obtain

$$\begin{aligned} \text{(B.1')} \quad & v_a(x_a^*) = \frac{(1-\varepsilon)(1+\delta)^2}{\alpha} \geq v_a(x_a^0) \text{ and} \\ & u(x_b^*, x_a^*) = U\left(\varepsilon(1 + \delta), \frac{(1-\varepsilon)(1+\delta)^2}{\alpha}\right) \geq U(v_b(x_b^1), v_a(x_a^1)). \end{aligned}$$

Note from the construction of the U function and (B1'), $U\left(\varepsilon(1 + \delta), \frac{(1-\varepsilon)(1+\delta)^2}{\alpha}\right) = U\left(\frac{(1-\varepsilon)(1+\delta)^2}{\alpha}, \varepsilon(1 + \delta)\right) \geq U(v_b(x_b^1), v_a(x_a^1))$. Moreover, note that $\varepsilon(1 + \delta) > \frac{1}{\alpha+1}$ holds, since $\varepsilon > \frac{1}{\alpha+1}$ by $\alpha > \frac{1-\varepsilon}{\varepsilon}$.

Next, by the exactly same argument as that developed in the proof of Theorem 1, it follows from **IR** of generation $t = 0$ that

$$\text{(B.2)} \quad (v_b(x_b^1), v_a(x_a^0)) \leq \left(\frac{(1-\varepsilon)(1+\delta)^2}{\alpha}, \frac{(1-\varepsilon)(1+\delta)^2}{\alpha}\right).$$

Then, by the exactly same argument as that developed in the proof of Theorem 1, construct an intertemporal feasible allocation $(x_a^0; (x'^t, l'^t, \omega_a'^t)_{t=1, \dots}; (y'^t)_{t=1, \dots}) \in F(E^\infty; \omega_a^0)$ with $(x_b^1, x_a^0) = \left(\left(\frac{\alpha}{\alpha+1}, \frac{1}{\alpha+1}\right), \left(\frac{1}{\alpha+1}, \frac{\alpha}{\alpha+1}\right)\right)$. Then, since $v_b(x_b^1) = \frac{1}{\alpha+1}$, $v_b(x_b^1) \leq \frac{(1-\varepsilon)(1+\delta)^2}{\alpha} < \frac{1}{\alpha+1}$ for the sufficiently small $\delta > 0$, and $\frac{1}{\alpha+1} < \varepsilon(1 + \delta)$ as observed above, we obtain by the construction of the monotonic function U ,

$$\text{(B.3')} \quad U(v_b(x_b^1), v_a(x_a^1)) < U(v_b(x_b^{1'}), v_a(x_a^1)) = U(v_b(x_b^{1'}), v_a(x_a^{1'})).$$

Moreover, we also have:

$$\text{(B.4')} \quad v_a(x_a^0) \leq \frac{(1-\varepsilon)(1+\delta)^2}{\alpha} < \frac{1}{\alpha+1} = v_a(x_a^{0'}).$$

Thus, by (B.3') and (B.4'), the φ -optimal allocation $(x_a^0; (x^t, l^t, \omega_a^t)_{t=1, \dots}; (y^t)_{t=1, \dots})$ cannot be weakly Pareto efficient, a contradiction. ■

Figure 1

