Vol. 83 No. 1 *The Keizai Gaku, Annual Report of the Graduate School of Economics and Management, Tohoku University* October 2026

Equilibrium and Traverse in a Continuous-Time Disintegrated Neo-Austrian Approach

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Abstract

Following John Hicks's seminal work *Capital and Time*, subsequent research has strived to extend his Neo-Austrian framework to multi-sectoral models by incorporating it into the Sraffian system. Although these extensions – known as the "disintegrated Neo-Austrian approach" – provide a general model of production processes, they have not yet been equipped with a rigorous analytical foundation. In light of this gap, the present study first provides a formal proof of the existence of equilibrium prices and quantities in a continuous-time multi-sectoral model, which fully incorporates both circulating and fixed capital through continuous-input-continuous-output time profiles. It then turns to traverse analysis as an application of this equilibrium analysis. Focusing on a technical change in a non-basic consumption-good sector, the study identifies new aspects of the traverse process. In particular, when a sufficiently output-enhancing technical change occurs, a situation can arise in which the output of the consumption good exceeds the non-innovation reference level, while the investment remains curtailed from that level due to a shortage of means of production—a situation characterised by Karl Marx as "double crisis."

1. Introduction

The Neo-Austrian approach refers to a stream of economic theory initiated by John Hicks's seminal work *Capital and Time* (1973) which is characterized by its modelling of the time-consuming nature of production processes through time profiles of inputs and outputs as follows :

- Time-consuming production involving both circulating and fixed capital is represented by continuous-input-continuous-output time profiles.
- The production period is divided into a construction period and a utilization period for fixed capital.
- Time is assumed to be continuous.
- New production processes start at any point in time as long as they are feasible.
- Even after a technical change, pre-innovation production processes already in operation coexist with post-innovation processes as long as they remain profitable.

Subsequent research following Hicks (1973) has endeavoured to extend this framework to multi-

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sectoral models by incorporating it into the Sraffian system (Belloc 1980, 1996 ; Baldone 1996)[1]. Although these extensions, which are referred to as the "disintegrated Neo-Austrian approach," attempt to achieve a level of generality comparable to that of the von Neumann model, it has been also argued that the Neo-Austrian approach — whether its integrated or disintegrated variant — can be reduced to a particular case of the von Neumann or Sraffian models (Burmeister 1974 ; Hagemann and Kurz 1976). Critics argue that production processes involving fixed and circulating capital can be more generally modelled in terms of joint production rather than time profiles, thereby rendering the Neo-Austrian approach redundant. This critique is valid. However, the situation would change once one departs from the conventional assumption of discrete time and instead adopts a continuous-time framework. As the dimensionality of input and output matrices is finite, it is impossible to represent continuous-time movements of both circulating and fixed capital solely in terms of joint production. From a mathematical standpoint, functions defined over discrete time are inherently continuous with respect to time and cannot capture any discontinuous movements pertaining to production processes. Therefore, the Neo-Austrian time-profile approach cannot be completely reduced to the joint-production framework of the von Neumann or Sraffian models.

That said, although the disintegrated (multi-sectoral) Neo-Austrian approach provides a highly general modelling of production processes, it has not yet been equipped with a rigorous analytical foundation : in particular, no formal proof of the existence of equilibrium prices and quantities has been established. Considering this gap, the present study aims to demonstrate the existence of a price and quantity equilibrium in a multi-sectoral model with continuous-input-continuous-output time profiles in continuous time. Section 2 and 3 will deal with the basic system (consisting of basic goods only) and a basic-non-basic system (including both basic and non-basic goods), respectively.

Next, as an application of the equilibrium analysis developed in Section 3, Section 4 turns to traverse analysis, distinguishing between short-term and long-term traverses on the one hand, and between activity paths and output paths on the other. Hicks (1973) may be regarded as a pioneering contribution in this regard, as it initiated the analysis of transitional dynamics ("traverse") following a technical change, based on the distinction between the construction period and the utilization period of fixed capital[2]. As noted above, Neo-Austrian traverse analysis is characterized by the temporary coexistence of pre-innovation and post-innovation processes, until the former complete their lifetimes. Previous traverse analyses have been conducted exclusively within the *integrated* (one-good) Neo-Austrian model (Hicks 1973 ; Belloc 1980 ; Nardini 1990, 1993) and may reasonably be expected to extend to technical changes in a *basic* sector within a multi-sectoral framework. In the present study, however, the traverse analysis focuses on a technical change in a *non-basic* consumption-good sector, aiming to uncover new aspects of the traverse process.

This endeavour could be further justified by recent historical-philological studies (Mori, 2018a, 2018b, 2019), which suggest that, in anticipation of the 1857 crisis, Karl Marx advanced the hypothesis of

1) See Gehrke (1998) on incompatibilities of the Neo-Austrian postulates with the general multi-sectoral analysis.

2) Traverse processes after distributional changes were studied in a three-sector model by Lowe (1976) (see Hagemann and Kurz, 1998).

a "double crisis," namely the simultaneous and interconnected occurrence of *overproduction* in the industrial market (consumption-goods) and *underproduction* in the produce market (raw materials). Marx ascribed this phenomenon to a disproportionate allocation between fixed and circulating capital in the post-innovation phase of the consumption-goods industry. However, it is not immediately clear why and how overproduction of consumption goods could arise in a situation characterized by shortages of raw materials, nor at what point in time and through which mechanisms such a discontinuity as was identified by Marx as a "crisis" might occur. The traverse analysis developed in this study may thus contribute to assessing the logical possibility of Marx's "double crisis" hypothesis. Finally, Section 5 concludes the study by summarizing the main results and identifying the remaining issues for future research.

2. Equilibrium in the Basic System

2.1. Technical Parameters

As mentioned above, there have been some attempts to generalise the Neo-Austrian model to "disintegrated Neo-Austrian approach" by integrating it to a Sraffian multi-sectoral structure (Belloc 1980, 1996 ; Baldone 1996)[3]. As far as we see, however, there has not been an endeavour which provides this generalised framework with an analytical foundation by proving the existence of a price and quantity equilibrium as we will do in this section.

To begin with, we define the technical parameters which are used in this model as follows. Let the number of sectors (and also goods, see the definition $b_{ij}(\tau)$ below) as $n \in \mathbb{R}_{++}$:

$D_j \in \mathbb{R}_{++}$: duration of production period for Sector j, which is technically determined by the physical life time of production processes in Sector j.

$d_j \in (0, D_j)$: duration of construction period for Sector j

i.e., $D_j - d_j$ is the duration of utilisation period for Sector j

$a_{ij}(\tau) \in \mathbb{R}_{+}$: input coefficient of good i for the production process in Sector j which started τ units of time ago with $\forall j \exists i\, a_{ij}(0) > 0$ (i.e., for every sector, some input is used at the beginning of each production period).

$l_j(\tau) \in \mathbb{R}_{+}$: labour coefficient for the production process in Sector j which started τ units of time ago with $l_j(0) > 0$.

$b_{ij}(\tau) \in \mathbb{R}_{+}$: output coefficient of good i for the production process in Sector j which started τ units of time ago. We assume a single production, i.e., $b_{ij}(\tau) = 0$ for all $i \neq j$ and all τ, whereby no output is produced during the construction period, i.e., $b_{jj}(\tau) = 0$ for all j and all $\tau \in [0, d_j)$, and some positive output is yielded at the beginning of each utilization period, i.e., $b_{jj}(d_j) > 0$ for all j.

Furthermore, $a_{ij}(\tau)$ and $l_j(\tau)$ are each defined as a function of τ which is differentiable in $(0, D_j)$ and one-sided differentiable at the endpoints and has the support $[0, D_j]$ (i.e., outside this interval, the function values vanish), whereas $b_{ij}(\tau)$ is differentiable in (d_j, D_j) and one-sided differentiable at the

3) See Gehrke (1998) on incompatibilities of the Neo-Austrian postulates with the general multi-sectoral analysis.

endpoints and has the support $[d_j, D_j]$[4].

Finally, for each of the above three kinds of coefficients, we defined a matrix as follows.

$$A(\tau) := (a_{ij}(\tau)) \in \mathbb{R}_+^{n\times n}$$

$$B(\tau) := (b_{ij}(\tau)) \in \mathbb{R}_+^{n\times n}$$

$$L(\tau) := (l_j(\tau)) \in \mathbb{R}_+^{1\times n}$$

This study will use the Laplace-transform for each technical coefficient in the following sections. For that purpose, let $\mathfrak{L}(f)$ denote the Laplace transform of a function f, so that for all $s \in \mathbb{C}$

$$\mathfrak{L}(a_{ij})(s) = \int_{-\infty}^{\infty} a_{ij}(\tau)e^{-s\tau}d\tau = \int_0^{D_j} a_{ij}(\tau)e^{-s\tau}d\tau$$

$$\mathfrak{L}(b_{ij})(s) = \int_{-\infty}^{\infty} b_{ij}(\tau)e^{-s\tau}d\tau = \int_0^{D_j} b_{ij}(\tau)e^{-s\tau}d\tau$$

$$\mathfrak{L}(l_j)(s) = \int_{-\infty}^{\infty} l_j(\tau)e^{-s\tau}d\tau = \int_0^{D_j} l_j(\tau)e^{-s\tau}d\tau.$$

Then, we define the "*elementwise* Laplace transform" of matrices $A(\tau)$, $B(\tau)$ and $L(\tau)$ as follows,

$$A^{\mathfrak{L}}(s) := (\mathfrak{L}(a_{ij})(s)) \in \mathbb{C}^{n\times n}$$

$$B^{\mathfrak{L}}(s) := (\mathfrak{L}(b_{ij})(s)) \in \mathbb{C}^{n\times n}$$

$$L^{\mathfrak{L}}(s) := (\mathfrak{L}(l_j)(s)) \in \mathbb{C}^{1\times n}$$

Finally, define the "*normalised* elementwise Laplace transform" of $A(\tau)$ and $L(\tau)$ as follows :

$$\bar{A}^{\mathfrak{L}}(s) = \left(\bar{a}_{ij}^{\mathfrak{L}}(s)\right) := \left(\frac{\mathfrak{L}(a_{ij})(s)}{\mathfrak{L}(b_{ij})(s)}\right) \in \mathbb{C}^{n\times n}$$

$$\bar{L}^{\mathfrak{L}}(s) = \left(\bar{l}_j^{\mathfrak{L}}(s)\right) := \left(\frac{\mathfrak{L}(l_j)(s)}{\mathfrak{L}(b_{jj})(s)}\right) \in \mathbb{C}^{1\times n},$$

2.2. Price Equilibrium

To consider the basic system, we make the following indecomposability assumption for input matrices.

Assumption 1. $A^{\mathfrak{L}}(0)$, *i.e.,* $\left(\int_0^{D_j} a_{ij}(\tau)d\tau\right)$ *is indecomposable.*

4) This differentiability assumption for time profiles is made solely for technical convenience. The results in the following analysis can be extended to more general functions, including discontinuous functions and delta-functions, which would be necessary for a full-scale treatment of fixed capital. For delta-functions and distributions see Zemanian (1965).

Remark 1. Assumption 1 implies the indecomposablity of both $A^{\mathfrak{L}}(r)$ and $\bar{A}^{\mathfrak{L}}(r)$ for all $r \in \mathbb{R}$. This result implies in turn that there are indeed basic goods in the system so that we are not dealing here with a genuine "Neo-Austrian model," which can ultimately be reduced to a one-good model, but a kind of hybrid with the Neumann-Sraffian multi-sectoral type (see Burmeister 1974 ; Hagemann and Kurz 1976).

Next, let's make sure of the following result.

Lemma 1. $\lim_{r\to\infty} \bar{a}_{ij}^{\mathfrak{L}}(r) = \infty$ *for some (i, j).*

Proof. By definition we have

$$\lim_{\tau\to 0+} a_{ij}(\tau) = a_{ij}(0) > 0 \text{ for some } (i, j) \text{ and}$$

$$\lim_{\tau\to 0+} b_{jj}(\tau) = b_{jj}(0) = 0 \text{ for all } j.$$

According to the Abelian theorem (Zemanian 1965, theorem 8.6-1, 243), we obtain

$$\lim_{r\to\infty} r\mathfrak{L}(a_{ij})(r) = a_{ij}(0) > 0 \text{ some } (i, j) \text{ and}$$

$$\lim_{r\to\infty} r\mathfrak{L}(b_{jj})(r) = b_{jj}(0) = 0 \text{ for all } j.$$

Therefore, for some (i, j)

$$\lim_{r\to\infty} \frac{\mathfrak{L}(a_{ij})(r)}{\mathfrak{L}(b_{jj})(r)} = \lim_{r\to\infty} \frac{r\mathfrak{L}(a_{ij})(r)}{r\mathfrak{L}(b_{jj})(r)} = \infty \quad . \text{(qed)}$$

Furthermore, we assume that the technique is productive.

Assumption 2. $\bar{A}^{\mathfrak{L}}(0) = \left(\dfrac{\int_0^{D_j} a_{ij}(\tau)d\tau}{\int_0^{D_j} b_{jj}(\tau)d\tau} \right)$ is productive, i.e., $\lambda(\bar{A}^{\mathfrak{L}}(0)) < 1$,

where $\lambda(M)$ denotes the Frobenius root of a non-negative square matrix M.

Then, we obtain the following result.

Lemma 2. *There exists a positive number* $\bar{r}$ *so that*

$$\lambda(\bar{A}^{\mathfrak{L}}(\bar{r})) = 1 \tag{1}$$

$$\textit{and } 0 < \lambda(\bar{A}^{\mathfrak{L}}(r)) < 1 \text{ for all } r \in [0, \bar{r}) \tag{2}$$

Proof. (1) follows from the continuity of the Laplace transform (Zemanian 1965, theorem 8.3-2, 225) and the Frobenius root, Lemma 1, Assumption 1, Remark 1, Assumption 2 and the intermediate value theorem. Next, define a set R_λ by $R_\lambda := \{r > 0 \mid \lambda(\bar{A}^{\mathfrak{L}}(\tau)) = 1\}$. Again, from the continuity of the

Laplace transform and the Frobenius root, R_λ is closed. Therefore, there exists the minimum of R_λ, which satisfies also (2). (qed)

As far as discrete time is assumed, the Neo-Austrian model could be revealed as a particular case of the von Neumann - Sraffa model (see Burmeister 1974 ; Hagemann and Kurz 1976), where the former's internal rate of return corresponds to the latter's rate of profit. The product price consists of input prices, each multiplied by a compounded profit factor $(1+r)^\tau$, where r is the rate of profit (or internal rate of profit) per unit of time, and τ denotes time difference between input and output. This applies to the present model with continuous time, where input price will be multiplied by the profit factor $e^{r\tau}$. Therefore, the price equation takes the following form.

$$p_j \int_0^{D_j} b_{jj}(\tau) e^{(D_j-\tau)r} d\tau = \sum_{i=1}^n p_i \int_0^{D_j} a_{ij}(\tau) e^{(D_j-\tau)r} d\tau + w \int_0^{D_j} l_j(\tau) e^{(D_j-\tau)r} d\tau \text{ for } j = 1,..., n \qquad (3)$$

$$\text{or } p_j \int_0^{D_j} b_{jj}(\tau) e^{-r\tau} d\tau = \sum_{i=1}^n p_i \int_0^{D_j} a_{ij}(\tau) e^{-r\tau} d\tau + w \int_0^{D_j} l_j(\tau) e^{-r\tau} d\tau \text{ for } j = 1,..., n \qquad (3')$$

where p_j, w and r denote the price of good j, the wage rate and the rate of profit, respectively. The left-hand side (LHS) of (3) represents the total output price while the first term of the right-hand side (RHS) is the total physical input price, and the second term is the total labour input price (wage), each evaluated at the end of production period D_j. Therefore, the price of inputs and outputs which took place τ units of time after the beginning of production period are discounted by $e^{r\tau}$. Equation (3') represents the price equation evaluated at the beginning of production period. Note that the wage rate w is assumed to be a positive constant throughout this paper. It is easy to confirm that the integral in each term of the price equation (3') is the Laplace transform of $b_{ij}(\tau)$, $a_{ij}(\tau)$ or $l_j(\tau)$. In fact, it has the economic meaning as the present value of an input or output flow, evaluated at the beginning of production period. Therefore, by using our notations defined in Section 2.1, we finally obtain the following price equation system.

$$pB^{\mathfrak{L}}(r) = pA^{\mathfrak{L}}(r) + wL^{\mathfrak{L}}(r) \ ; \ \text{i.e.,} \quad p = p\bar{A}^{\mathfrak{L}}(r) + w\bar{L}^{\mathfrak{L}}(r) \qquad (4)$$

Proposition 1. *There exist a positive profit rate r and a positive price vector p which satisfy the price equation (4).*

Proof. Since $\bar{A}^{\mathfrak{L}}(r)$ is indecomposable for all $r \in \mathbb{R}$ (by Assumption 1 and Remark 1), and $I - \bar{A}^{\mathfrak{L}}(r)$ fulfils Hawking-Simon condition for all $r < \bar{r}$ (by Lemma 2), we have $(I - \bar{A}^{\mathfrak{L}}(r))^{-1} > 0$. Take any $r \in (0, \bar{r})$ and $p := w\bar{L}^{\mathfrak{L}}(r)(I - \bar{A}^{\mathfrak{L}}(r))^{-1} > 0$, and these r and p satisfy (4). (qed)

We call any positive rate of profit r and positive price vector p satisfying the price equation (4) an equilibrium rate of profit and equilibrium price, respectively.

2.3. Balanced Growth

Having considered the price system in the previous section, let's move on to quantity system, in

particular, equilibrium activity and equilibrium growth rate in a balanced growth. Let $c \in \mathbb{R}_+^{n\times 1}$ denote a *semipositive* consumption vector for workers. We ignore non-workers' consumption (or "take-out").

Balanced Growth refers to a property of the quantity system characterised by the same rate of growth of activity (and hence output) across all sectors. Let $x := (x_1,..., x_n)$ be the activity vector at time 0 and g be the rate of growth, and the activity vector at time t will be xe^{gt}. Then, considering all production processes operating at time t, the total output in Sector j at time t amounts to the following.

$$\int_0^{D_j} b_{ij}(\tau)x_j e^{(t-\tau)g}d\tau = x_j e^{gt}\int_0^{D_j} b_{ij}(\tau)e^{-g\tau}d\tau = x_j e^{gt}\mathfrak{L}(b_{ij})(g)$$

Similarly, the total physical input and total labour input are as follows.

$$\int_0^{D_j} a_{ij}(\tau)x_j e^{(t-\tau)g}d\tau = x_j e^{gt}\int_0^{D_j} a_{ij}(\tau)e^{-g\tau}d\tau = x_j e^{gt}\mathfrak{L}(a_{ij})(g)$$

$$\int_0^{D_j} l_j(\tau)x_j e^{(t-\tau)g}d\tau = x_j e^{gt}\int_0^{D_j} l_j(\tau)e^{-g\tau}d\tau = x_j e^{gt}\mathfrak{L}(l_j)(g)$$

From this, it is easy to understand that the Laplace transform has a second economic meaning besides expressing the present value of input or output flow as was shown in the previous section. Here, the Laplace transform of input or output time profile represents the total amount of input or output at time 0 on a balanced growth path. Therefore, the equilibrium condition between supply and demand at time t is as follows :

$$B^{\mathfrak{L}}(g)xe^{gt} = A^{\mathfrak{L}}(g)xe^{gt} + \frac{w}{pc}cL^{\mathfrak{L}}(g)xe^{gt}$$

$$B^{\mathfrak{L}}(g)x = A^{\mathfrak{L}}(g)x + \frac{w}{pc}cL^{\mathfrak{L}}(g)x, \tag{5}$$

where p denotes the equilibrium price as was defined in Proposition 1. Then, we obtain the following result.

Proposition 2. *For any c defined above and $w > 0$, there exist a positive growth rate g and a positive activity vector x which satisfy the equilibrium condition (5) and $g = r$, where r denotes an equilibrium profit rate determined in Proposition 1.*

Proof. By Proposition 1, for any $w > 0$, there exist a positive profit rate r and positive price vector p which satisfy

$$p = pM(r), \tag{6}$$

where $M(r)$ is defined by

$$M(r) := \left[\bar{A}^{\mathfrak{L}}(r) + \frac{w}{pc}c\bar{L}^{\mathfrak{L}}(r)\right].$$

By the indecomposability of $\bar{A}^{\mathfrak{L}}(r)$ for all r, the matrix $M(r)$ is indecomposable. And the equation (6) implies $\lambda(M(r)) = 1$. Therefore, from the Perron-Frobenius theorem, it follows that there exists a positive vector z associated with r which satisfies

$$z = M(r)z = \bar{A}^{\mathfrak{L}}(r)z + \frac{w}{pc}c\bar{L}^{\mathfrak{L}}(r)z \tag{7}$$

Let $x = (x_j) := \left(\frac{z_j}{b_{jj}(r)} \right)$, and this x and r satisfy (5). (qed)

3. Equilibrium in a Basic-Non-Basic System

In the previous section, we considered the basic system, where all goods are basic based on Assumption 1. In particular, the assumption implies that any consumption good also serves as means of production somewhere. As a particular case of the system including both basic and non-basic goods, we now assume consumption goods in a genuine sense that they are not used as means of production, and consider input and output matrices with the following partitions.

$$A(\tau) := \begin{pmatrix} A_{11}(\tau) & A_{12}(\tau) \\ 0 & 0 \end{pmatrix}; \quad A_{11}(\tau) \in \mathbb{R}_+^{m\times m}, A_{12}(\tau) \in \mathbb{R}_+^{m\times(n-m)}$$

$$B(\tau) := \begin{pmatrix} B_{11}(\tau) & 0 \\ 0 & B_{22}(\tau) \end{pmatrix}; \; B_{11}(\tau) \in \mathbb{R}_+^{m\times m}, B_{22}(\tau) \in \mathbb{R}_+^{(n-m)\times(n-m)}$$

$$L(\tau) := (L_1(\tau) \quad L_2(\tau)); L_1(\tau) \in \mathbb{R}_+^{m}, L_2(\tau) \in \mathbb{R}_+^{(n-m)}$$

The input and output matrices of Laplace transforms are partitioned accordingly.

$$A^{\mathfrak{L}}(s) := \begin{pmatrix} A_{11}^{\mathfrak{L}}(s) & A_{12}^{\mathfrak{L}}(s) \\ 0 & 0 \end{pmatrix}$$

$$B^{\mathfrak{L}}(s) := \begin{pmatrix} B_{11}^{\mathfrak{L}}(s) & 0 \\ 0 & B_{22}^{\mathfrak{L}}(s) \end{pmatrix}$$

$$L^{\mathfrak{L}}(s) := (L_1^{\mathfrak{L}}(s) \quad L_2^{\mathfrak{L}}(s))$$

$$\bar{A}^{\mathfrak{L}}(s) := \begin{pmatrix} \bar{A}_{11}^{\mathfrak{L}}(s) & \bar{A}_{12}^{\mathfrak{L}}(s) \\ 0 & 0 \end{pmatrix}$$

$$\bar{L}^{\mathfrak{L}}(s) := (\bar{L}_1^{\mathfrak{L}}(s) \quad \bar{L}_2^{\mathfrak{L}}(s))$$

Correspondingly, the price vector is partitioned as $p = (p_1, p_2)$.

Obviously, Sectors $m+1$ to n are non-basic (as genuine consumption goods) and do not enter any sector as inputs, and Sectors 1 to m are assumed to be basic, and $A_{11}^{\mathfrak{L}}(0)$ to be productive. For short, Assumption 1 and 2 should now apply to $A_{11}^{\mathfrak{L}}(0)$ and $\bar{A}_{11}^{\mathfrak{L}}(0)$, respectively. Then, Proposition 1 implies immediately the following corollary.

Corollary 1. *There exist a positive profit rate r and a positive price vector p_1 which satisfy the following condition :*

$$p_1 = p_1 \bar{A}_{11}^{\mathfrak{L}}(r) + w\bar{L}_1^{\mathfrak{L}}(r)$$

And from these r and p_1, a positive price vector for non-basic goods p_2 follows uniquely, i.e.,

$$p_2 = p_1 \bar{A}_{12}^{\mathfrak{L}}(r) + w\bar{L}_2^{\mathfrak{L}}(r)$$

Remark 2. Obviously, the prices of basic goods p_1 and the profit rate r depend solely on the basic sectors' technical coefficients, i.e., $\bar{A}_{11}^{\mathfrak{L}}(s)$ and $\bar{L}_1^{\mathfrak{L}}(s)$, and wage rate w. This implies that the technical coefficients in non-basic sectors, i.e., $\bar{A}_{12}^{\mathfrak{L}}(s), \bar{L}_2^{\mathfrak{L}}(s)$, and therefore the prices of non-basics p_2 have no influence on p_1 nor r.

Let's move on to the quantity system, in particular, a balanced growth in the basic-non-basic system defined above. For that purpose, we need to redefine the consumption vector for workers c as follows.

$$c := \begin{pmatrix} 0 \\ c_2 \end{pmatrix} \in \mathbb{R}_+^{n\times 1}; \quad c_2 \in \mathbb{R}_{++}^{(n-m)\times 1}$$

Then, the equilibrium condition in the quantity system also needs to be rewritten as follows.

$$B^{\mathfrak{L}}(g)xe^{gt} = A^{\mathfrak{L}}(g)xe^{gt} + \frac{w}{p_2 c_2} cL^{\mathfrak{L}}(g)xe^{gt}$$

$$\text{or } B^{\mathfrak{L}}(g)x = A^{\mathfrak{L}}(g)x + \frac{w}{p_2 c_2} cL^{\mathfrak{L}}(g)x \tag{5'}$$

where p_2 denotes the equilibrium price of (genuine) consumption goods as was determined in Corollary 1. Then, we obtain the following result.

Corollary 2. *For any c defined above and $w > 0$, there exist a positive growth rate g and a positive activity vector x which satisfy the equilibrium condition (5') and $g = r$, where r denotes an equilibrium profit rate determined in Corollary 1.*

Proof. By Corollary 1, for any $w > 0$, there exist a positive profit rate r and positive price vectors p_1 and p_2 which satisfy

$$p = (p_1 \quad p_2) = pM(r), \tag{6'}$$

where $M(r)$ is defined by

$$M(r) := \left[\bar{A}^{\mathfrak{L}}(r) + \frac{w}{p_2 c} c\bar{L}^{\mathfrak{L}}(r)\right] = \begin{pmatrix} \bar{A}_{11}^{\mathfrak{L}}(r) & \bar{A}_{12}^{\mathfrak{L}}(r) \\ \dfrac{w}{p_2 c_2} c_2 \bar{L}_1^{\mathfrak{L}}(r) & \dfrac{w}{p_2 c_2} c_2 \bar{L}_2^{\mathfrak{L}}(r) \end{pmatrix}.$$

By the indecomposability of $\bar{A}_{11}^{\mathfrak{L}}(r)$ and the definition of A, L and c, the matrix $M(r)$ is indecomposable. And the equation (6') implies $\lambda(M(r)) = 1$. Therefore, from the Perron-Frobenius theorem, it follows that there exists a positive vector z associated with r which satisfies

$$z = M(g)z = \bar{A}^{\mathfrak{L}}(g)z + \frac{w}{p_2 c}c\bar{L}^{\mathfrak{L}}(g)z \tag{7'}$$

Let $x = (x_j) := \left(\frac{z_j}{b_{jj}(r)}\right)$, and this x and r satisfy (5'). (qed)

Obviously, (7') can be rewritten as follows.

$$z_1 = \bar{A}_{11}^{\mathfrak{L}}(g)z_1 + \bar{A}_{12}^{\mathfrak{L}}(g)z_2 \tag{8}$$

$$z_2 = \frac{w}{p_2 c_2}c_2\bar{L}_1^{\mathfrak{L}}(g)z_1 + \frac{w}{p_2 c_2}c_2\bar{L}_2^{\mathfrak{L}}(g)z_2 \tag{9}$$

Assume now $n = 2$ and $m = 1$, i.e., that there is only one basic and only one non-basic sector. Let the growth rate g $(=r)$ be given as determined in Corollary 2 (and Corollary 1) and let the basic sector's output z_1 (>0) be arbitrarily given. Then, with z_2 which can be uniquely determined by (8), $z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$ fulfils (9) automatically. Needless to mention: under the assumption of $n - m = 1$, the scalar c_2 is cancelled out so that it does not play any role. Obviously, $e^{gt}z = e^{gt}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$ is a balanced-growth *output* path, and $e^{gt}x = e^{gt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ is a balanced-growth *activity* path.

4. Traverse after a Technical Change in Sector II

4.1. Definitions and Assumptions

The multi-sectoral traverse analysis is in general highly complicated and mathematically quite demanding (Belloc 1980, 1996). We need to simplify the model in order to capture the essential properties of the traverse. It may justify analysing the traverse process by means of a simple two-sector model. In the following, the economy is assumed to consist of two sectors, where Sector I and II produce respectively means of production and means of consumption (as genuine consumption goods). Then, Sector I is basic and Sector II is non-basic because the means of production as output of Sector I goes into both sectors as physical input, whereas the means of consumption as output of Sector II goes to neither sector as physical input.

It is assumed that before time 0, the economy is in an equilibrium state where price equilibrium and balanced growth are realised in the sense of Corollaries 1 and 2. At time 0, a technical change occurs in Sector II where at least one of the technical parameters belonging to Sector II changes. In the following, each technical parameter before time 0 is denoted by starring the corresponding symbol: e.g. a^*_{ij} (τ) before 0 and a_{ij} (τ) after 0. It is natural to assume that a new technique can be adopted only if it would be profitable, i.e., it would make an extra-profit based on the current price and profit rate.

Assumption 3. A new technique is adopted only if

$p_2 = p_1\bar{A}_{12}^{*\mathfrak{L}}(r) + w\bar{L}_2^{*\mathfrak{L}}(r) > p_1\bar{A}_{12}^{\mathfrak{L}}(r) + w\bar{L}_2^{\mathfrak{L}}(r)$ holds.

As stated in Remark 2, the basic good's price p_1 and the profit rate r remain unchanged throughout the traverse because the technical parameters of Sector I remain unchanged. On the other hand, the non-basic good's price p_2 may change, whereby there is assumed to be no truncation of production process in either sector. As for the activity level, it is natural to assume that Sector I continues with the balanced growth path because the profitability perspective remains unchanged throughout the traverse (recall that the profit rate r is unaffected). Let $x_1(t)$ and $x_2(t)$ denote the activity level of Sector I and II respectively. Then, the assumption states as follows.

Assumption 4. For some positive constant ξ_1, the Sector I's activity level, $x_1(t)$, fulfils $x_1(t) = \xi_1 e^{gt}$ for all t, where g denotes the balanced-growth rate before time 0 as determined in Corollary 2.

On the other hand, the investment is supposed to become more active in Sector II since the introduction of a new profitable technique at time 0 in the sense of Assumption 3. Therefore, Sector II is assumed to invest as much as the provision of means of production from Sector I allows.

Assumption 5. Sector II's activity level, $x_2(t)$, fulfils for all t

$$\int_{\min\{t,0\}}^{\min\{t,D_2\}} a_{12}(\tau)x_2(t-\tau)d\tau + \int_{\max\{0,t\}}^{\max\{t,D_2^*\}} a_{12}^*(\tau)x_2(t-\tau)d\tau = \int_0^{D_1^*}(b_{11}^*(\tau) - a_{11}^*(\tau))x_1(t-\tau)d\tau. \quad (10)$$

Note that $a_{11}(\tau) = a_{11}^*(\tau)$, $b_{11}(\tau) = b_{11}^*(\tau)$ and $D_1 = D_1^*$ hold because Sector I's technique does not change. The LHS represents Sector II's demand for means of production at time t, consisting of the demand from the new production processes (the first term) and that from the old production processes (the second term). The RHS represents the means of production which Sector I provides for Sector II at time t after deducting an amount for self-use from its total output. Therefore, Equation (10) implies the full employment of means of production by both sectors at every time. Note that the first term of the LHS vanishes at all $t<0$ because the new technique is not available yet, whereas the second term vanishes at all $t > D_2^*$ because the old technique is no longer used anywhere.

Considering Assumption 4 and the fact that balanced growth is realised before $t < 0$, we have from (8) and $x_j = \dfrac{z_j}{b_{jj}(g)}$ (for j=1,2)

$$x_1(t) = \xi_1 e^{gt} \quad \text{for all t}$$

$$x_2(t) = \xi_2 e^{gt} \quad \text{for } t < 0,$$

where we define ξ_2 by $\xi_2 := \dfrac{\mathfrak{L}(b_{11}^*)(g) - \mathfrak{L}(a_{11}^*)(g)}{\mathfrak{L}(a_{12}^*)(g)}\xi_1$, and g denotes again the balanced-growth rate before time 0 as determined in Corollary 2. Obviously, (ξ_1, ξ_2) is a solution of (5'). In the following, we normalise the activity levels by setting $\xi_2 = 1$ and define a factor k by $k := (\mathfrak{L}(b_{11}^*)(g) - \mathfrak{L}(a_{11}^*)(g))\xi_1$ or $k := \mathfrak{L}(a_{12}^*)(g)$ so that the right-hand side of (10) can be written as follows :

$$\int_0^{D_1}(b_{11}^*(\tau)-a_{11}^*(\tau))x_1(t-\tau)d\tau = ke^{gt} = \mathfrak{L}(a_{12}^*)(g)e^{gt} = a_{12}^* * (e^{gt}), \tag{11}$$

where * denotes the convolution of two functions (or distributions).

4.2. The Traverse Path in the Long Run

4.2.1. Activity level

First, we examine the long-run properties of the activity level on the traverse path. The activity level throughout the traverse, $\bar{x}_2(t)$, can be determined by (10) as follows.

$$\begin{aligned} &a_{12} * \bar{x}_2 + a_{12}^* * \underline{x}_2^* = ke^{gt} \\ &\text{or } h := a_{12} * \bar{x}_2 = ke^{gt} - a_{12}^* * \underline{x}_2^*, \end{aligned} \tag{12}$$

where $\bar{x}_2$ and $\underline{x}_2^*$ are defined as follows.

$$\bar{x}_2(t) := \begin{cases} x_2(t) & \text{for } t \geq 0 \\ 0 & \text{for } t < 0 \end{cases}$$

$$\text{and } \underline{x}_2^* := \begin{cases} 0 & \text{for } t \geq 0 \\ e^{gt} & \text{for } t < 0 \end{cases}.$$

Note that h is defined as the convolution of two functions, a_{12} and $\bar{x}_2$, and that it is a function of t with $h(t)$ representing the demand for means of production from Sector II's new production processes.

From (11) and (12) we have :

$$h = a_{12} * \bar{x}_2 = a_{12}^* * (e^{gt}) - a_{12}^* * \underline{x}_2^* = a_{12}^* * \bar{x}_2^* \tag{13}$$

where $\bar{x}_2^*$ is defined by

$$\bar{x}_2^* := \begin{cases} e^{gt} & \text{for } t \geq 0 \\ 0 & \text{for } t < 0 \end{cases}.$$

From (13), we obtain the following properties of h :

$$h(t) = 0 \qquad \text{for } t \leq 0 \tag{14.1}$$

$$h(t) = ke^{gt} \quad \text{for } t \geq D_2^* \tag{14.2}$$

$$h'(t) = 0 \qquad \text{for } t < 0 \tag{14.3}$$

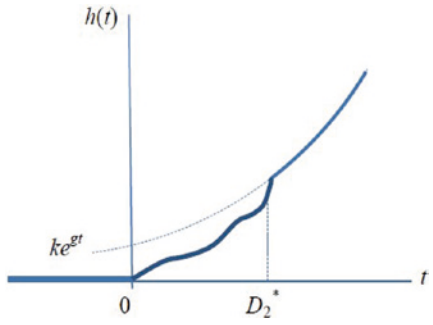
$$h'(t) = g(a_{12}^* * \bar{x}_2^*) + a_{12}^*(t) \quad \text{for } t \in (0, D_2^*) \tag{14.4}$$

$$h'(t) = \frac{d(ke^{gt})}{dt} = gke^{gt} \quad \text{for } t > D_2^* \tag{14.5}$$

$$\mathfrak{L}(h)(s) = \mathfrak{L}(a_{12}^*)(s)\mathfrak{L}(\bar{x}_2^*)(s) = \frac{\mathfrak{L}(a_{12}^*)(s)}{s-g}, \tag{14.6}$$

which yield such a graph as depicted in Figure 1. From these properties of h, we can obtain the following result.

Figure 1. Graph of $h(t)$



Proposition 3. *The activity level of Sector II, $x_2(t)$ for $x \geq 0$, is the first derivative in the distributional sense of a continuous function with $x_2(t) \sim e^{gt}$ as $t \to \infty$.*

See proof in Appendix. The first distributional derivative of a continuous function may have discontinuous points (jumps). However, in the sense of $\lim_{t\to\infty} \frac{x_2(t)}{e^{gt}} = \gamma$ (constant), $x_2(t)$ "converges in the long run to a steady-growth path with the growth rate g which is Sector I's growth rate and the balanced-growth rate prevailing before time 0.

4.2.2. Output

Next, we examine the long-run path of output in Sector II throughout the traverse process. If enough long time passed so that t is larger than D^*_2, the output path thereafter can be traced by following the activity path emerging after the introduction of the new technology, i.e., $\bar{x}_2$. Then, the output at time t, $B_2(t)$, can be written as

$$B_2(t) := \int_0^t b_{22}(\tau) x_2(t-\tau) d\tau = b_{22} * \bar{x}_2, \tag{15}$$

where $\bar{x}_2$ was introduced in (12). It is easy to imagine that the output path $B_2(t)$ would follow the activity path in the long run and "converge as well toward a steady-growth path with the old balanced-growth rate g. The next proposition verifies this intuition.

Proposition 4. *The output path of Sector II, $B_2(t)$, is a continuous function of t with $B_2(t) \sim e^{gt}$ as $t \to \infty$.*

The proof follows in Appendix.

4.3. The Traverse Path in the Short Run

4.3.1. Activity level

Next, we examine the short-run properties of the traverse path. The first production process starting after the technical change at time 0 ends at time D_2. Within this period, the traverse path is supposed to perform a peculiar movement. To avoid complications a general analysis could cause and to

achieve clear-cut results, we need to introduce a specific assumption and specify the time profile which would otherwise allow a vast range of possibilities.

A study on the production process requiring duration of time was intensively conducted by Marx in his theory of capital turnover in *Capital*, Volume II. Using a number of numerical examples (a total of 22 examples in his manuscript) (see Mori 2006), he scrutinised the influence of the length of turnover on production scale (i.e., activity level), capital requirement and surplus. In all his numerical examples, he assumed a constant input during the production process in order to be able to focus on the genuine effects of time duration.

Assumption 6. The time profiles of input and output are each a step function in the manner that

$$a_{ij}(t) := \alpha_{ij}(H(t) - H(t - D_j))$$

$$l_j(t) := \lambda_j(H(t) - H(t - D_j))$$

$$b_{ij}(t) := \beta_{ij}(H(t - d_j) - H(t - D_j)),$$

where α_{ij}, β_{ij} and λ_j are positive numbers and $H(t)$ is the Heaviside function : i.e.,

$$H(t) = \begin{cases} 1 & (t \geq 0) \\ 0 & (t < 0) \end{cases}.$$

All pre-innovation technical parameters, i.e., a_{ij}^*, b_{ij}^* and l_j^*, are assumed in the same manner with respectively α_{ij}^*, β_{ij}^* and λ_j^* being their positive coefficient, and d_j^* and D_j* their length of construction period and production period, respectively. Furthermore, $D_j > D_j$* is assumed in the following analysis. Without any loss of generality, we normalise the unit of production process starting after time 0 by setting $\alpha_{ij} = \alpha_{ij}^*$.

First, we examine the short-run properties of the *activity level* on the traverse path. Let's start again with the function $h = a_{12} * \bar{x}_2$ defined in (13) and its properties (14.1) - (14.6). Take the Laplace-transform of h and use Assumption 6 to get :

$$\begin{aligned}
\mathfrak{L}(\bar{x}_2)(s) &= \frac{\mathfrak{L}(h)(s)}{\mathfrak{L}(a_{12})(s)} = \frac{\mathfrak{L}(h)(s)}{\frac{\alpha_{12}}{s}(1 - e^{-sD_2})} \\
&= \frac{1}{\alpha_{12}} \sum_{i=0}^{\infty} s e^{-sD_2 i} \mathfrak{L}(h)(s) \\
&= \frac{1}{\alpha_{12}} \sum_{i=0}^{\infty} \mathfrak{L}(\delta^{(1)}(t - D_2 i))(s) \mathfrak{L}(h)(s) \\
&= \frac{1}{\alpha_{12}} \sum_{i=0}^{\infty} \mathfrak{L}(\delta^{(1)}(t - D_2 i) * h)(s) \\
&= \frac{1}{\alpha_{12}} \sum_{i=0}^{\infty} \mathfrak{L}(h^{(1)}(t - D_2 i))(s)
\end{aligned}$$

Then, we obtain the following activity path.

$$\bar{x}(t) = \frac{1}{\alpha_{12}} \sum_{i=0}^{\infty} h^{(1)}(t - D_2 i) \tag{16}$$

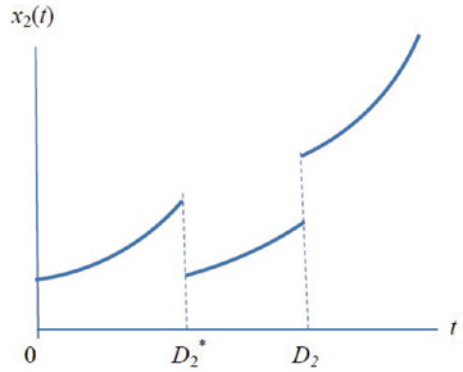
where $\delta^{(1)}$ and $h^{(1)}$ stand for the first derivative in the distributional sense of the delta function and the continuous function h respectively. As the properties (14.1) to (14.5) suggest, h is not differentiable everywhere so that $h^{(1)}$ has discontinuous points, i.e., an upward jump at time 0 to a steady-growth path $\alpha_{12}{}^{*} e^{gt}$ and a downward jump at time $D_2{}^{*}$ to a lower steady-growth path gke^{gt}. The activity path (16) shows that the same pattern is successively introduced in the cycle of D_2 time units (see Figure 2). The result can be summarised in the following proposition.

Proposition 5. *After following the old balanced-growth path e^{gt} until time $D_2{}^{*}$, the activity path $\bar{x}_2(t)$ undergoes at time $D_2{}^{*}$ a vertical drop on the scale of 1 to a lower steady-growth path $\frac{gk}{\alpha_{12}} e^{gt}$ and instantly recovers at time D_2 on the scale of 1 to a new steady-growth path. The drop on the scale of 1 recurs periodically at time $D_2{}^{*}$, $D_2+D_2{}^{*}$, $2D_2+D_2{}^{*}$, $3D_2+D_2{}^{*}$,..., and the path instantly recovers every time after the passage of $D_2-D_2{}^{*}$ time units on the scale of 1 to a new steady-growth path.*

Remark 3. Proposition 5 implies that the difference in the production period between the old and new technique, $D_2-D_2{}^{*}$, determines the length of periodical curtailment of activity level. As the production period has been extended, the number of simultaneously operating production processes increased, which occupy so much means of production for themselves that new investment has to be temporarily curtailed. A further implication is that due to Assumptions 5 and 6, the activity level cannot drop to a minus during the period from $D_2{}^{*}$ to D_2 so that the so-called "Hayek effect" does not occur (see Zamagni 1984).

In the above discussion in this section, it is implicitly assumed that the new technique has a longer lifetime than the old one. It is indeed plausible that a mechanisation process tends to prolong the pro-

Figure 2. The short-run activity path after time 0



duction period because of a longer construction period.

4.3.2. Output

Next, we examine the short-run properties of the *output* path $B_2(t)$ in the traverse process. To begin with, it is necessary to specify the time structure for the new technique compared to the old one. As mentioned above, compared to the manufacture, it seems to be characteristic of the technical progress by mechanisation that both the construction period and the production period as a whole are prolonged.

Assumption 7. Through the technical change, the time structure of production process changes as follows :

$$d_2 > d_2^*$$
$$D_2 > D_2^*$$

According to Assumption 7, it is useful to divide the first production period, which starts at time 0, into three intervals, i.e., $[0, d_2^*]$, $[d_2^*, d_2]$ and $[d_2, D_2]$. In the first interval, some old production processes and all new processes are still in the construction period and yield no output yet. The second interval is characterised by difference in the construction period between the new and old technique, where all old production processes and no new processes are in the utilisation period. In the third interval, some new production process is working in the utilisation process.

Utilising Assumption 6 and Proposition 5, we can calculate the output path up to time D_2 as follows :

$$B_2(0) = \frac{\beta_{22}^* \sigma_1}{g} \tag{17}$$

$$B_2(d_2^*) = \frac{\beta_{22}^* \sigma_1}{g} e^{g d_2^*} \tag{18}$$

$$B_2(d_2) = \begin{cases} \dfrac{\beta_{22}^*}{g} e^{gd} (\sigma_1 - \sigma_3) & \text{for } d_2 \le D_2^* \\ 0 & \text{for } d_2 \ge D_2^* \end{cases} \tag{19}$$

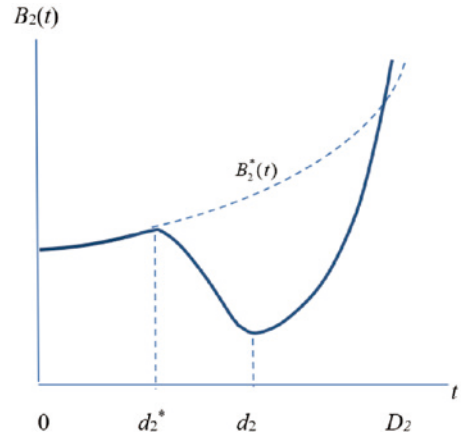
$$B_2(D_2^*) = \begin{cases} 0 & \text{for } D_2^* \le d_2 \\ \dfrac{\beta_{22} e^{g D_2^*}}{g} (\sigma_2 - \sigma_4) & \text{for } D_2^* \ge d_2 \end{cases} \tag{20}$$

$$B_2(D_2) = \begin{cases} \dfrac{\beta_{22} \sigma_2}{g} e^{g D_2} & \text{for } D_2 \le D_2^* + d_2 \\ \dfrac{\beta_{22} \sigma_5}{g} e^{g D_2} & \text{for } D_2 \ge D_2^* + d_2 \end{cases} \tag{21}$$

where we define

$$\sigma_1 := e^{-g d_2^*} - e^{-g D_2^*}$$

Figure 3. The short-run output path after time 0



$$\sigma_2 := e^{-gd_2} - e^{-gD_2}$$

$$\sigma_3 := e^{-gd_2^*} - e^{-gd_2}$$

$$\sigma_4 := e^{-gD_2^*} - e^{-gD_2}$$

$$\sigma_5 := e^{-gd_2} - e^{-g(D_2^*+d_2)}$$

The output path $B_2(t)$ can be depicted with the graph in Figure 3. It turns downward at time d_2^* because an old process finishes its lifetime at every time while no new process can enter the utilisation period. As from time d_2 new processes begin to provide output, the path begins to rise again.

Now, let $x_2^*(t) := \underline{x}_2^*(t) + \bar{x}_2^*(t) = e^{gt}$ (for all t) be the reference activity path, i.e., the activity path which would be realised if the technical change had never occurred. Then, we compare the output between the reference activity path and the real activity path ($x_2(t) := \underline{x}_2^*(t) + \bar{x}_2(t)$ for all t), and in doing this, we can evaluate the change of supply position in consumption good per consumer (employee). For that purpose, we make the following restrictive assumption in order to concentrate on the effect of the time duration : the technical change is assumed to be "Hicks-neutral," where the capital-labour ratio (technical composition of capital) is not affected by a technical change. To sum up :

Assumption 8. $\alpha_{ij}/\lambda_j = \alpha_{ij}^*/\lambda_j^*$ holds.

As we are assuming the normalisation $\alpha_{ij} = \alpha_{ij}^*$, Assumption 8 implies $\lambda_j = \lambda_j^*$.

Remark 4. Assumption 4, 5 and 8 imply that the employment continues to increase on the same steady-growth path with the growth rate of g throughout the whole time.

Then, we can establish that at time D_2, the real output level, $B_2(D_2)$, is larger than the level the reference path would achieve with the old technique, $B_2^*(D_2) := (b_{22}^* * x_2^*)(D_2)$, if β_{22} is large enough. It can be formulated more precisely as follows.

Proposition 6. *$B_2(D_2) > B_2^*(D_2)$ if and only if $\beta_{22} > \theta\beta_{22}^*$, where θ is defined by* $\theta := \max\left\{\frac{\sigma_1}{\sigma_5}, \frac{\sigma_1}{\sigma_2}\right\}$.

See proof in Appendix. Obviously, the threshold $\theta\beta_{22}^*$ will be lower as d_2 becomes smaller. And in case of $D_2 - d_2 \leq D_2^*$, i.e., in case that the new utilisation period is not longer than the old production period as a whole, $B_2(D_2) > B_2^*(D_2)$ is always true because Assumption 3 guarantees its validity.

5. Conclusion

This study (Propositions 1 and 2) establishes a formal proof of the existence of equilibrium prices and a balanced-growth path in a multi-sectoral Neo-Austrian model with continuous time, fully incorporating both circulating and fixed capital through continuous-input-continuous-output time profiles. However, the lifetime of fixed capital — and thus the duration of the production process — is assumed to be exogenously given. This assumption is restrictive, as the lifetime of fixed capital is usually regarded as being endogenously determined (e.g., truncated) by the rate of profit and relative prices. Future research should extend the model to allow for such endogeneity.

Proposition 6 implies that if a technical change occurs which sufficiently enhances output as is represented by a large β_{22}, a situation emerges where in a neighbourhood of time D_2 before the end of the first post-innovation production process, the production of the consumption good *exceeds* the non-innovation reference level (see Figure 3) although in the meantime the investment ("rate of start") remains curtailed from that level because of the *shortage* of means of production (see Figure 2).

The results of our analysis (Proposition 5) also show that discontinuities emerge periodically along the post-innovation activity path, despite steady growth in the supply of raw materials and labor. Marx might have been interested in such a jump of activity level at time D_2 (see Figure 2) where the raw material market is abruptly released from the continuously lasting pressure, which would mean a slump of raw material prices (Marx 2017, 519-521 ; Mori 2018a, 216).

Incidentally, a sharp contraction of output $B_2(t)$ ("gross product"), first theoretically identified by Ricardo (1951) as a consequence of the construction of new machinery, was confirmed in integrated traverse analyses (Hicks 1976 ; Zamagni 1984). The present study shows that this phenomenon likewise emerges in a multi-sectoral framework (see the interval $[d_2^*, d_2]$ in Figure 3).

Because of the high complexity of traverse problems, however, we needed some simplifying assumptions such as Marx time profiles (Assumption 6) and Hicks neutrality of technical changes (Assumption 7). Relaxing these assumptions should be addressed in future studies.

(Accepted 6 July 2026)

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Appendix

A.1. Proof of Proposition 3

For the proof of this proposition, as we only need to concentrate on Sector II, we will omit for the sake of simplification, subscripts indicating Sector II.

We have from (14.1), (14.3) and (14.4) :

$$\lim_{t\to 0+}\frac{h(t)}{t}=\lim_{t\to 0+}\frac{h(t)-h(0)}{t}=\lim_{t\to 0+}h'(t)=a^{*}(0)=:\gamma_1>0,$$

which implies by the Abelian theorem (Zemanian 1965, theorem 8.6-1, 243) :

$$\lim_{s\to\infty}\frac{s^2\mathfrak{L}(h)(s)}{2}=\gamma_1 \tag{A.1}$$

On the other hand, we have due to the Abelian theorem :

$$\lim_{t\to 0+}a(t)=a(0)>0,$$

which implies :

$$\lim_{s\to\infty}s\mathfrak{L}(a)(s)=a(0) \tag{A.2}$$

Now, we Laplace-transform both sides of (12) as follows :

$$\mathfrak{L}(a)(s)\mathfrak{L}(\bar{x})(s)=\mathfrak{L}(h)(s)$$

$$\mathfrak{L}(\bar{x})(s)=\frac{\mathfrak{L}(h)(s)}{\mathfrak{L}(a)(s)}$$

Then, we obtain from (A.1) and (A.2)

$$\lim_{s\to\infty}s\mathfrak{L}(\bar{x})(s)=\lim_{s\to\infty}\frac{s^2\mathfrak{L}(h)(s)}{s\mathfrak{L}(a)(s)}=\frac{2\gamma_1}{a(0)},$$

which implies that $\bar{x}(t)$ is the first derivative of a continuous function (Zemanian 1965, theorem 8.4-1, 236).

Furthermore, we can show from (12) and (14.6) :

$$\lim_{s\to 0+}s\mathfrak{L}(\bar{x}(t)e^{-gt})(s)=\lim_{s\to 0+}\frac{s\mathfrak{L}(h)(s+g)}{\mathfrak{L}(a)(s+g)}=\frac{\mathfrak{L}(a^*)(g)}{\mathfrak{L}(a)(g)}=:\gamma_2>0, \tag{A.3}$$

which implies again from the Abelian theorem (Zemanian 1965, theorem 8.7-1, 249) :

$$\lim_{s\to\infty}\bar{x}(t)e^{-gt}=\gamma_2 \quad \text{(qed)}$$

A.2. Proof of Proposition 4

The continuity follows immediately from the definition (15) and Proposition 3. Next, we Laplace-transform both sides of (15) to obtain

$$\mathfrak{L}(B_2)(s)=\mathfrak{L}(b_{22})(s)\mathfrak{L}(\bar{x}_2)(s)$$

Considering (A.3) we have :

$$\lim_{s\to 0+}s\mathfrak{L}(B_2e^{-gt})(s)=\lim_{s\to 0+}\mathfrak{L}(b_{22})(s+g)s\mathfrak{L}(\bar{x}_2e^{-gt})(s)=\mathfrak{L}(b_{22})(g)\gamma_2=:\gamma_3>0,$$

which implies from the Abelian theorem :

$$\lim_{s\to\infty}B_2(t)e^{-gt}=\gamma_3 \quad \text{(qed)}$$

A.3. Proof of Proposition 6

We calculate the output on the reference activity path as follows :

$$B_2^*(t) := b_{22}^* * x^* = e^{gt}\,\frac{\beta_{22}^*\sigma_1}{g} \quad \text{for all } t$$

$$B_2^*(D_2) = e^{gD_2}\,\frac{\beta_{22}^*\sigma_1}{g}$$

(i) Case of $D_2 - d_2 \leq D_2^*$

Obviously, $\theta = \dfrac{\sigma_1}{\sigma_2}$ is true in this case. The proposition follows directly by considering (21).

(ii) Case of $D_2 - d_2 \geq D_2^*$

We have $\theta = \dfrac{\sigma_1}{\sigma_5}$ in this case. Then, we establish the proposition from (21) again. (qed)