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Discussion Paper No. 491

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November 2024 (This version: January 2025)

TOHOKU ECONOMICS RESEARCH GROUP

Discussion Paper

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Stabilizing Structural Change and Cycle: Goodwin Meets Neumann on the Turnpike

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Abstract

This study examines the applicability of the turnpike property in nonlinear optimal control theory to economics. The turnpike property implies that, under certain assumptions, the optimal path, which is the solution to the dynamically optimal control problem, remains for almost all periods close to the Neumann ray (i.e., turnpike), which is the solution to the static optimal control problem, if sufficiently long periods are considered. We consider structural economic dynamics and cycles as examples of applicable fields. Based on Goodwin's growth cycle, we construct a dynamically nonlinear optimal control model using government cost function. Our numerical analysis demonstrates that the optimal path remains near the turnpike for almost all periods if sufficiently long periods are considered, in contrast to Goodwin's assertion that the turnpike embedded in the growth cycle model is unstable. Because the essence of the turnpike property is unrelated to the number of sectors, it can be applied to a broader class of models that have rarely been considered in the analysis of the turnpike theorem. We also show the possibility of the government choosing the turnpike (i.e., the ideal trajectory) by determining the parameters and form of the cost function. This study develops on the theoretical implications for Pasinetti's work on structural economic dynamics to show that *institutions* matter for stable economic growth.

Keywords: Structural change; Growth cycle; Nonlinear optimal control; Exponential turnpike; Neumann ray; Cost function; Riccati equation.

JEL: C61; E32; E37; E62.

1. Introduction

Structural change and (growth) cycle are research fields in economics pioneered by the Cambridge Keynesians. Luigi Pasinetti (1981, 1993) created a new research field, called structural economic dynamics, in the 1960s. He considered economic growth accompanied by continuous changes in the structure (or composition) of prices, output, and employment over time. As Pasinetti (1993, p. 1) rightly pointed out, "[t]he term 'structural economic dynamics' was, until a short time ago, practically unknown in the economic literature." This is because many economists had focused intensively on the existence of and convergence toward a steady (balanced) growth path, typically represented by Solow (1956) and Neumann (1945).

Pasinetti's structural economic dynamics is a *pre-institutional* analysis in that it focuses on the essential futures of structural economic dynamics, independently of any institutional setup, in industrialized economies (Pasinetti, 2007). By contrast, Goodwin analyzed structural economic dynamics and growth cycle in capitalist economies (e.g., Goodwin, 1982, 1983, 1986, 1989, 1990a, 1990b, 1993; Goodwin and Punzo, 1987; Landesmann and Goodwin, 1994). A characteristic of his approach is the application of nonlinear dynamics in economics, allowing us to investigate phenomena to which neoclassical economists rarely pay attention. For example, Goodwin (1967, p. 58) applied the competition between predator and prey, following Lotka (1925) and Volterra (1931), in describing the Marxian "contradiction of capitalism and its transitory resolution in booms and slums" and demonstrated the occurrence of the endogenously perpetual growth cycles,[1] which goes against convergence to a steady growth path. He persistently emphasized nonlinearity and instability in understanding irregular growth in capitalist economies.

Furthermore, Goodwin (1982, 1990a, 1990b, 1993) analyzed the effects of government behavior on the stability of capitalist economies. Goodwin (1990a, 1990b, 1993) deserve attention because Rössler's dynamic model was applied. The model is globally stabilized, but free to perform wildly erratic motion locally around the equilibrium owing to the existence of the dynamic control parameter.[2] Furthermore, he showed that the amplitude of the erratic fluctuations in the employment rate and wage share was drastically reduced by varying the dynamic control parameter, which can be regarded as the government's employment rate target.

In this study, we examine the applicability of the turnpike property in nonlinear optimal control theory in economics by considering structural economic dynamics/growth cycle as an example of an applicable field. Goodwin (1990a, 1990b, 1993) also focused on the effects of the government's behavior on economic performance, the difference between Goodwin's and our approach is that unlike in Goodwin's model, the government's behavior is based on the optimization method in our approach. A moderate assumption of contemporary economic theory is that government intervenes optimally.[3]

Turnpike is a well-known term in economics that refers to the Neumann (1945) ray of steady growth. We consider economic programming to guide an economic system from the initial to the final position, both of which are exogenously given. The turnpike theorem implies that, under certain assumptions, a growth path starting from the initial position will remain near the turnpike if the programming period is sufficiently large.

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[1] Samuelson (1972a, 1972b) were also interested in the Lotka-Volterra model.

[2] See Di Matteo and Sordi (2015) on Goodwin's application of the Rössler's dynamic model to economics.

[3] See, for example, Kydland and Prescott (1977).

According to McKenzie (1998), Samuelson (1949) was the first to discover this property. Dorfman et al. (1958) inspired many to elaborate the proof of the turnpike theorem under various assumptions.

In linear control theory, independent of economics, the turnpike property has been studied as an *exponential dichotomy* (e.g., Wilde and Kokotovic, 1972). It means that the norm of the projection onto the stable subspace of any orbit in the system decays exponentially in forward time ($t \to \infty$) and the norm of the projection onto the unstable subspace of any orbit decays exponentially in reverse time ($t \to -\infty$). It was generalized to nonlinear optimal control theory (e.g., Anderson and Kokotovic, 1987). Porretta and Zuazua (2013) formulated an exponential turnpike in the linear finite/infinite-dimensional optimal control theory, which was extended to the nonlinear finite-dimensional optimal control theory by Trélat and Zuazua (2015). The turnpike property is applied in biology, space missions, brain evolution in humans, design and understanding of machine learning methods, sharp optimization in aircraft design, membrane filtration systems, and control of chemical reactors with uncertain models (Faulwasser and Grüne, 2022).

The great advantage of applying nonlinear optimal control theory is that the applicable environments of the turnpike theorem are drastically widened. In this study, we applied nonlinear optimal control to a model in which structural changes/growth cycles could be caused by nonlinearity. Such a model has rarely been considered in the arguments of the turnpike theorem. The turnpike property in the nonlinear optimal control theory analyzes the relationship between the dynamically optimal path and the steady growth path. Like ours, some studies have investigated the relationship between structural changes and the steady growth path under "structural change and Kaldor's facts" (Kurose, 2021). Since all the models used in the research are based on Ramsey (1928), the turnpike property holds, as argued by Intriligator (2002). By contrast, we investigate the relationship between structural change and the steady growth path based on Goodwin (1967). Every trajectory in Goodwin's (1967) model is a closed orbit (except for the non-trivial fixed points) and generates oscillations. Such a model has never been examined in relation to the turnpike theorem. However, nonlinear optimal control theory allows us to investigate the relationship between the dynamic path and the steady growth path.

Indeed, Goodwin (1986) argued that the oscillations generated by the growth cycle are "swinging along the turnpike," where the turnpike means the (nontrivial) fixed point in Goodwin's (1967) model and corresponds to the Neumann ray. Furthermore, he stated that the turnpike in Goodwin (1967) was unstable because the path starting from an arbitrary initial position never reached it. We demonstrate that the government's optimal behavior can substantially reduce the "swing" and converges to the turnpike, if sufficiently long periods are given. This result is comparable to Goodwin's (1990a,1990b, 1993), which also considers the stabilizing effects of government activity on economic performance.

The essence of the turnpike property is unrelated to the dimension of the state and control variables. Examining the *applicability* of the turnpike property to economics does not require an elaborate model. Goodwin (1967) is the model that can be disaggregated, as in Goodwin (1986, 1989) and Punzo and Goodwin (1987). If we can demonstrate the validity of the turnpike theorem using the Goodwin (1967) model, extending it to generate structural changes and cycle, the general applicability of the turnpike theorem in nonlinear optimal control model to economics can be verified.

The remaining paper is organized as follows. Section 2 presents a review of the turnpike theorem in finite-dimensional nonlinear optimal control theory. Section 3 presents the optimal control model based on Goodwin (1967) and conducts a numerical analysis showing that the turnpike properties holds. Section 4 discusses the implications of the numerical analysis and Section 5 concludes.

2. Turnpike Theorem and Nonlinear Optimal Control Theory

2.1. The turnpike theorem in economics

The turnpike theorem, as well as the paradoxes of capital theory, were intensively debated in economic theory in the 1960s and 1970s. The theorem states that under certain assumptions, an optimal growth path starting from an arbitrary initial position stays near the efficient steady growth path (Neumann ray), except for the terminal position, if a sufficiently long period is considered.

In the proof of this theorem, beginning with Dorfman et al. (1958), many economists focused on closed models, indicating neither an inflow of goods from the outside into the system nor any outflow of goods from the system. In the models, all goods (even labor) reproduce themselves within the system using the Leontief, Neumann, and smooth production technology. Morishima (1961), Radner (1961), McKenzie (1963a, 1963b, 1963c), Inada (1964), and Nikaido (1964) are the early works representative of this model. They sought to prove that the optimal path maximizing social preferences at the terminal position, starting from the given initial position, remains close to the Neumann ray, given a sufficiently long period. This type of analysis is called the *final state turnpike* (McKenzie, 1971). Although there are differences in the models, all proofs have fundamentally the same features, as summarized in Morishima (1969, pp. 184–195).

The proofs of the turnpike theorem were presented in the open growth models as well, in which labor is an inflow and consumption is an outflow. This type of analysis, called the *consumption turnpike*, is based on the Ramsey (1928) model. Atsumi (1965), Gale (1967), Koopmans (1965), Samuelson (1965), Cass (1966), Tsukui (1967), McKenzie (1968), and Morishima (1969, Chap. 13) are the early works using this model. They proved the consumption turnpike theorem, asserting the proximity of optimal growth paths to a particular steady growth path along which the maximum utility level of per capita consumption is maintained, given a sufficiently long period. Furthermore, the consumption turnpike theorem is more difficult to prove when the discounting factor is positive, and the relationship between the growth rate and discount factor has also received attention (e.g., Morishima, 1969; McKenzie, 2008). McKenzie (1998, 2008) has provided a comprehensive survey of this theorem.

Almost all proofs of the theorem have been made under restrictive assumptions, such as convexity and free disposability. The exception is Dechert and Nishimura (1983), who demonstrated the validity of the theorem in a model in which convexity is not always satisfied. They assumed that the production function exhibits increasing returns in the early phase but decreasing returns in the terminate phase in the Ramsey model with only one good. Another exception is Khan and Piazza (2011), who demonstrated the validity of the theorem without assuming free disposability.

The turnpike theorem is realistic, although its proof is highly mathematical. Chakravarty (1969) examined the possibility of applying this theorem to developmental planning. Tsukui (1968), Tsukui and Murakami (1979), and Móczár and Tsukui (1995) applied it to empirical research in Japan using input-output tables; Tsukui's (1968) model was used for economic planning by the Japanese government (Yoshioka and Kawasaki, 2016).

Recently, many natural science fields have focused on the turnpike property of optimal control theory because it allows us to know, at least approximately, the optimal trajectory without solving the dynamic optimal problem if the turnpike theorem holds. It must be confirmed whether the conditions for the turnpike property are satisfied. The turnpike property is particularly relevant where dynamic optimal control problems are extremely complicated because of nonlinearity; therefore, it can be applied to vast areas of science, as mentioned in Section 1.

2.2. Turnpike property in finite-dimensional nonlinear optimal control

Given $T > 0$, consider the dynamically nonlinear optimal control problem (**DOC**) to determine the control variables denoted by the vector $\boldsymbol{u}_T(\cdot) \in L^\infty([0,T];\mathbb{R}^m)$ and the corresponding state variables denoted by $\boldsymbol{x}(t) \in \mathbb{R}^n$.

$$\textbf{DOC}:\ \min_{\boldsymbol{x}(\cdot),\boldsymbol{u}(\cdot)} C_T\big(\boldsymbol{x}(t),\boldsymbol{u}(t)\big) = \min_{\boldsymbol{x}(\cdot),\boldsymbol{u}(\cdot)} \int_0^T f^0\big(\boldsymbol{x}(t),\boldsymbol{u}(t)\big)dt,$$

$$\text{s.t.}\, \dot{\boldsymbol{x}}(t) = f\big(\boldsymbol{x}(t),\boldsymbol{u}(t)\big), \qquad (1)$$

$$R\big(\boldsymbol{x}(0),\boldsymbol{x}(T)\big) = \boldsymbol{0}, \qquad (2)$$

where $f: \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is of class C^2, $R: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^k$ is a mapping of class C^2, and $f^0: \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ is of a function of class C^2. C_T is the cost function of the **DOC** for $t \in [0,T]$, and R represents the initial and terminal conditions to be satisfied.

The Hamiltonian function $\mathcal{H}$ for the **DOC** is given by

$$\mathcal{H}\big(\boldsymbol{x}(t),\boldsymbol{\lambda}(t),\lambda^0,\boldsymbol{u}(t)\big) = \lambda^0 f^0\big(\boldsymbol{x}(t),\boldsymbol{u}(t)\big) + \boldsymbol{\lambda}^\intercal(t) f\big(\boldsymbol{x}(t),\boldsymbol{u}(t)\big), \qquad (3)$$

where $\boldsymbol{\lambda}(t) \in \mathbb{R}^n_{++}$ denotes the vector of auxiliary variables and superscript $\intercal$ denotes the transpose.

According to Pontryagin's maximum principle, there exists continuous mapping $\boldsymbol{\lambda}_T(t) \in \mathbb{R}^n$ and $\lambda^0_T \in \mathbb{R}$, such that, for almost every $t \in [0,T]$,

$$\frac{dx^i_T(t)}{dt} = \frac{\partial\mathcal{H}\big(\boldsymbol{x}_T(t),\boldsymbol{\lambda}_T(t),\lambda^0,\boldsymbol{u}_T(t)\big)}{\partial\lambda^i},$$

$$\frac{d\lambda^i_T(t)}{dt} = -\frac{\partial\mathcal{H}\big(\boldsymbol{x}_T(t),\boldsymbol{\lambda}_T(t),\lambda^0,\boldsymbol{u}_T(t)\big)}{\partial x^i},$$

$$\frac{\partial\mathcal{H}\big(\boldsymbol{x}_T(t),\boldsymbol{\lambda}_T(t),\lambda^0,\boldsymbol{u}_T(t)\big)}{\partial u^j} = 0,$$

for $i = 1,\cdots,n$ and $j = 1,\cdots,m$. Without loss of generality, we can normalize $\lambda^0_T = -1$.

In addition, we require transversality conditions, which take several forms depending on the assumptions about the terminal conditions. Trélat and Zuazua (2015) generally formulated the conditions as follows. There exists $(\gamma_1,\cdots,\gamma_k) \in \mathbb{R}^k$ such that

$$\begin{pmatrix} -\boldsymbol{\lambda}_T(0) \\ \boldsymbol{\lambda}_T(T) \end{pmatrix} = \sum_{i=1}^{k} \gamma_i \nabla R_i\big(\boldsymbol{x}_T(0), \boldsymbol{x}_T(T)\big), \qquad (4)$$

where $\nabla R_i\big(\boldsymbol{x}_T(0), \boldsymbol{x}_T(T)\big) \equiv \begin{pmatrix} \frac{\partial R\left(x_T(0), x_T(T)\right)}{\partial x_i} \\ \frac{\partial R\left(x_T(0), x_T(T)\right)}{\partial y_i} \end{pmatrix}$.[4] The triplet $\big(x_T(t), \lambda_T(t), u_T(t)\big)$ is the solution to **DOC**.

The corresponding static optimal control problem (**SOC**) can be defined as

$$\begin{aligned} \textbf{SOC} \quad & \min_{\boldsymbol{x},\boldsymbol{u}} f^0(\boldsymbol{x}, \boldsymbol{u}), \\ \text{s.t.} \quad & f(\boldsymbol{x}, \boldsymbol{u}) = \boldsymbol{0}. \end{aligned}$$

SOC is a typical optimization problem constrained by nonlinear equalities. We assume that the **SOC** has at least one solution, denoted by $(\bar{\boldsymbol{x}}, \bar{\boldsymbol{\lambda}}, \bar{\boldsymbol{u}})$.

The Lagrangian function for the **SOC** is given by

$$\mathcal{L}(\boldsymbol{x}, \boldsymbol{\lambda}, \lambda_0, \boldsymbol{u}) = \lambda^0 f^0(\boldsymbol{x}, \boldsymbol{u}) + \boldsymbol{\lambda}^\intercal f(\boldsymbol{x}, \boldsymbol{u}).$$

The Lagrangian multiplier method implies that there exists $(\bar{\boldsymbol{\lambda}}, \bar{\lambda}^0)$ such that

$$\frac{\partial \mathcal{L}(\bar{\boldsymbol{x}}, \bar{\boldsymbol{\lambda}}, \bar{\lambda}^0, \bar{\boldsymbol{u}})}{\partial \lambda^i} = 0, \frac{\partial \mathcal{L}(\bar{\boldsymbol{x}}, \bar{\boldsymbol{\lambda}}, \bar{\lambda}^0, \bar{\boldsymbol{u}})}{\partial x^i} = 0, \frac{\partial \mathcal{L}(\bar{\boldsymbol{x}}, \bar{\boldsymbol{\lambda}}, \bar{\lambda}^0, \bar{\boldsymbol{u}})}{\partial u^j} = 0.$$

Without loss of generality, we can normalize $\bar{\lambda}^0 = -1$.

The turnpike property indicates the relationship between **DOC** and **SOC**. Roughly speaking, this suggests that if T is sufficiently large, the optimal path given by the **DOC** remains mostly close to the point given by **SOC** under certain assumptions. The solution to **SOC** is constant over time, implying that the relative ratios of the variables remain constant over time. In this context, the solution to **SOC** can be considered a turnpike.

We formally define the *exponential turnpike*.

Definition 1 – Exponential Turnpike (Trélat and Zuazua, 2015): *There exists* $\eta > 0$ *and* $C > 0$ *such that, for every* $T > 0$, *the* ***DOC*** *has the solution* $\big(\boldsymbol{x}_T(t), \boldsymbol{\lambda}_T(t), \boldsymbol{u}_T(t)\big)$ *satisfying*

$$\|\boldsymbol{x}_T(t) - \bar{\boldsymbol{x}}\| + \|\boldsymbol{\lambda}_T(t) - \bar{\boldsymbol{\lambda}}\| + \|\boldsymbol{u}_T(t) - \bar{\boldsymbol{u}}\| \leqq C\big(e^{-\eta t} + e^{-\eta(T-t)}\big) \quad (5)$$

for $t \in [0, T]$.

[4] If $k \leqq 2n$, the terminal condition must be imposed in the **DOC**. The typical cases are 1) both the initial and final positions are fixed; 2) the initial position is fixed but the final position is free; 3) the initial position is fixed and the final position is constrained by the inequalities $\boldsymbol{0} \leqq \boldsymbol{\varphi}\big(\boldsymbol{x}_T(T)\big)$: $\mathbb{R}^n \to \mathbb{R}^p$. Case 1) means that $\boldsymbol{x}(0) = \boldsymbol{x}_0$ and $\boldsymbol{x}(T) = \boldsymbol{x}_T$. Thus, $R(\boldsymbol{x} - \boldsymbol{x}_0, \boldsymbol{y} - \boldsymbol{y}_0) = \boldsymbol{0}$ holds. Function (2) gives us no information in this case. Case 2) means that $R(\boldsymbol{x}, \boldsymbol{y}) = \boldsymbol{x} - \boldsymbol{x}_0$ and the transversality condition is given by $\boldsymbol{\lambda}_T(T) = \boldsymbol{0}$. In Case 3) $R(\boldsymbol{x}, \boldsymbol{y}) = \big(\boldsymbol{x} - \boldsymbol{x}_0, \boldsymbol{\varphi}(\boldsymbol{y})\big)$ holds. Thus, the transversality condition is given by $\lambda_T^i(T) = \sum_{i=1}^{k} \gamma_i \frac{\partial \varphi_i(x_T(T))}{\partial x^i}$, where $k = n + p$. See, for example, Kamien and Schwartz (2000, pp. 155–163).

Inequality (5) implies that, except for the beginning and the end of the time interval $[0, T]$, the dynamic triplet $(\boldsymbol{x}_T(t), \boldsymbol{\lambda}_T(t), \boldsymbol{u}_T(t))$ is exponentially close to the static triplet $(\bar{x}, \bar{\lambda}, \bar{u})$. While Porretta and Zuazua (2013) also defined the exponential turnpike, the characteristic of the definition in Trélat and Zuazua (2015) is that it included auxiliary variables, as shown in (5), although Porretta and Zuazua (2013) defined the exponential turnpike without including them.

To understand the important concepts necessary to solve the optimal control problem, consider the case in which (1) has the following form.

$$\dot{\boldsymbol{x}}(t) = f(\boldsymbol{x}(t), \boldsymbol{u}(t)) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}\boldsymbol{u}(t), \quad (6)$$

where $\mathbf{A} \in \mathbb{R}^{n \times n}, \mathbf{B} \in \mathbb{R}^{n \times m}$.

Definition 2 – Controllability (Kwakernaak and Sivan, 1972, p. 54): *System (6) is controllable if, for any given initial position $\boldsymbol{x}_0$ and the final position $\boldsymbol{x}_T$, there exist $T > 0$ and a (piecewise continuous) control function $\boldsymbol{u}(t)$, for $t \in [0, T]$, such that the admissible solution $\boldsymbol{x}_u(t)$ with $\boldsymbol{x}_u(0) = \boldsymbol{x}_0$ satisfies $\boldsymbol{x}_u(T) = \boldsymbol{x}_T$.*

Controllability implies that the linear system can reach the final position within a specific time interval through admissible control. Kalman (1960) formulated the condition for (perfect) controllability based on (6) as follows.

Theorem 1 – Condition for Controllability (Kalman, 1960): *System (6) is (completely) controllable if and only if*

$$\mathrm{Rank}[\boldsymbol{B}, \boldsymbol{AB}, \cdots, \boldsymbol{A}^{n-1}\boldsymbol{B}] = n, \quad (7)$$

where $[\boldsymbol{B}, \boldsymbol{AB}, \cdots, \boldsymbol{A}^{n-1}\boldsymbol{B}] \in \mathbb{R}^{n \times mn}$ denotes the composite matrix.

Proof: See Kalman (1960). ■

The Kalman condition (7) implies the coverage of the entire state space by a controllable subspace, that is, the column vectors of matrix $[\boldsymbol{B}, \boldsymbol{AB}, \cdots, \boldsymbol{A}^{n-1}\boldsymbol{B}]$ span the n-dimensional space to ensure the controllability (Kwakernaak and Sivan, 1972, p. 55).

Let $D^2\mathcal{H}(\bar{\boldsymbol{x}}, \bar{\boldsymbol{\lambda}}, -1, \bar{\boldsymbol{u}})$ be the Hessian matrix of (3):

$$D^2\mathcal{H}(\bar{\boldsymbol{x}}, \bar{\boldsymbol{\lambda}}, -1, \bar{\boldsymbol{u}}) = \begin{bmatrix} \boldsymbol{\mathcal{H}}_{xx} & \boldsymbol{\mathcal{H}}_{x\lambda} & \boldsymbol{\mathcal{H}}_{xu} \\ \boldsymbol{\mathcal{H}}_{\lambda x} & \boldsymbol{0}_n & \boldsymbol{\mathcal{H}}_{\lambda u} \\ \boldsymbol{\mathcal{H}}_{ux} & \boldsymbol{\mathcal{H}}_{u\lambda} & \boldsymbol{\mathcal{H}}_{uu} \end{bmatrix} \in \mathbb{R}^{(2n+m)\times(2n+m)},$$

where $\boldsymbol{\mathcal{H}}_{xx} \equiv \frac{\partial^2\mathcal{H}(\bar{\boldsymbol{x}}, \bar{\boldsymbol{\lambda}}, -1, \bar{\boldsymbol{u}})}{\partial x^2}$, $\boldsymbol{\mathcal{H}}_{x\lambda} \equiv \frac{\partial^2\mathcal{H}(\bar{\boldsymbol{x}}, \bar{\boldsymbol{\lambda}}, -1, \bar{\boldsymbol{u}})}{\partial x \partial \lambda} \in \mathbb{R}^{n \times n}$, $\boldsymbol{\mathcal{H}}_{xu} \equiv \frac{\partial^2\mathcal{H}(\bar{\boldsymbol{x}}, \bar{\boldsymbol{\lambda}}, -1, \bar{\boldsymbol{u}})}{\partial x \partial \boldsymbol{u}}$, $\boldsymbol{\mathcal{H}}_{\lambda u} \equiv \frac{\partial^2\mathcal{H}(\bar{\boldsymbol{x}}, \bar{\boldsymbol{\lambda}}, -1, \bar{\boldsymbol{u}})}{\partial \lambda \partial \boldsymbol{u}} \in \mathbb{R}^{n \times m}$, $\boldsymbol{\mathcal{H}}_{uu} \equiv \frac{\partial^2\mathcal{H}(\bar{\boldsymbol{x}}, \bar{\boldsymbol{\lambda}}, -1, \bar{\boldsymbol{u}})}{\partial \boldsymbol{u}^2} \in \mathbb{R}^{m \times m}$, and $\boldsymbol{0}_n \in \mathbb{R}^{n \times n}$ is the null matrix. Note that $\boldsymbol{\mathcal{H}}_{\lambda x} = \boldsymbol{\mathcal{H}}_{x\lambda}^{\top}$, $\boldsymbol{\mathcal{H}}_{ux} = \boldsymbol{\mathcal{H}}_{xu}^{\top}$, and $\boldsymbol{\mathcal{H}}_{u\lambda} = \boldsymbol{\mathcal{H}}_{\lambda u}^{\top}$. The following theorem holds.

Theorem 2: *Assume that*

1) $\boldsymbol{\mathcal{H}}_{uu}$ *is a symmetric negative definite and non-singular matrix;*

2) $\boldsymbol{W} \equiv -\boldsymbol{\mathcal{H}}_{xx} + \boldsymbol{\mathcal{H}}_{xu}\boldsymbol{\mathcal{H}}_{uu}^{-1}\boldsymbol{\mathcal{H}}_{ux}$ *is a symmetric positive definite matrix;*
3) $\boldsymbol{A} \equiv \boldsymbol{\mathcal{H}}_{\lambda x} - \boldsymbol{\mathcal{H}}_{\lambda u}\boldsymbol{\mathcal{H}}_{uu}^{-1}\boldsymbol{\mathcal{H}}_{ux}$ *and* $\boldsymbol{B} \equiv \boldsymbol{\mathcal{H}}_{\lambda u}$ *satisfy (7);*
4) *Point* $(\bar{\boldsymbol{x}}, \bar{\boldsymbol{x}})$ *is not a singular point of the mapping* R *and the norm of the Hessian of* R *at* $(\bar{\boldsymbol{x}}, \bar{\boldsymbol{x}})$ *is sufficiently small. That is, a small* $\varepsilon > 0$ *exists such that*

$$\bar{D} \equiv \|R(\bar{\boldsymbol{x}}, \bar{\boldsymbol{x}})\| + \left\| \begin{pmatrix} -\bar{\boldsymbol{\lambda}} \\ \bar{\boldsymbol{\lambda}} \end{pmatrix} - \sum_{i=1}^{k} \gamma_i \nabla R_i(\bar{\boldsymbol{x}}, \bar{\boldsymbol{x}}) \right\| \leqq \varepsilon. \quad (8)$$

Then, there exist constants $C > 0, \eta > 0$ *and a time* $T_0 > 0$ *such that, for any* $T > T_0$*, the* ***DOC*** *has at least one optimal solution* $(\boldsymbol{x}_T(t), \boldsymbol{\lambda}_T(t), \boldsymbol{u}_T(t))$ *satisfying (5) for* $t \in [0, T]$.

Proof: See Trélat and Zuazua (2015). ■

Remark 1. Theorem 2 holds, even if the state equation does not have the form shown in (6). This theorem does not guarantee the uniqueness of the optimal solution. As the theorem addresses general nonlinear optimal control problems, the **DOC** may have other solutions that do not pass around the steady growth path (Trélat and Zuazua, 2015, p. 90).

Remark 2. Theorem 2 holds, regardless of whether the terminal position is left free or fixed, as shown by Trélat and Zuazua (2015).

Remark 3. Assumption (8) is termed the "smallness condition." It means that $R(\bar{\boldsymbol{x}}, \bar{\boldsymbol{x}}) \simeq \boldsymbol{0}$ and $\binom{-\bar{\lambda}}{\bar{\lambda}} \simeq \sum_{i=1}^{k} \gamma_i \nabla R_i(\bar{x}, \bar{x})$. Putting another way, $(\bar{\boldsymbol{x}}, \bar{\boldsymbol{\lambda}})$ must be *almost* a solution of (2) and (4). Inequality (8) implies that $\boldsymbol{x}_0$ and $\boldsymbol{x}_T$ must be close to $\bar{\boldsymbol{x}}$ if the initial and terminal positions are fixed. Similarly, it implies that $\boldsymbol{x}_0$ must be close to $\bar{\boldsymbol{x}}$ and $\|\bar{\boldsymbol{\lambda}}\|$ must be small enough if the terminal position is left free (Trélat and Zuazua, 2015, p. 89). Thus, (5) is local.

3. The Turnpike Property in the Model with Structural Economic Dynamics and Cycle

We construct a nonlinear optimal model with structural changes and growth cycles based on Goodwin's (1967) model. We introduce a cost function minimized by the government into the model and demonstrate that the turnpike property holds true.

Although Goodwin (1967) is an aggregate macro model, it can be easily disaggregated, as in Goodwin (1986, 1989) and Goodwin and Punzo (1987), by introducing sectoral differences in labor productivity growth rates and capital-output ratios. Sectoral differences cause structural economic dynamics so that each sector has a growth rate and thus, its own everlasting cycle. The number of sectors (dimensions of the state and control variables) is not related to the essential features of the turnpike property. As we aim to examine the applicability of the turnpike property in nonlinear optimal control theory to economics, we do not require an elaborate model. The 1967 model is adequate, and we can show a new insight into the relationship between the turnpike and the growth cycle.

3.1. Goodwin (1967) model

Goodwin made the following assumptions.

1) Steady (disembodied) technical progress: $a(t) \equiv {}^{Y(t)}\big/_{L(t)} = a_0 e^{\alpha t}$, where $a(t)$ and $\alpha > 0$ denote the labor productivity at time t and its growth rate.
2) Steady growth in the labor force N: $N(t) = N_0 e^{\beta t}$, where $\beta > 0$ denotes its growth rate.
3) Only two factors of production (labor and capital), both homogeneous and non-specific, exist.
4) All quantities are real and net.
5) All wages are consumed and all profits saved and automatically invested.
6) The capital-output ratio $\kappa \equiv {}^{K(t)}\big/_{Y(t)}$ is constant.
7) The real wage rate w rises in the neighborhood of full employment.

Let $x_1(t), x_2(t)$ be the wage share and the rate of employment, respectively.

$$x_1(t) = \frac{w(t)L(t)}{Y(t)} = \frac{w(t)}{a(t)}, \qquad x_2(t) \equiv \frac{L(t)}{N(t)},$$

where $L(t)$ denotes employment at time t. The structure of the model is summarized as follows.

$$\dot{x}_1(t) = x_1(t)\big(-(\alpha+\gamma) + \rho x_2(t)\big), \quad (9)$$

$$\dot{x}_2(t) = x_2(t)\left(\frac{1 - x_1(t)}{\kappa} - (\alpha+\beta)\right). \, (10)$$

The non-trivial fixed point is given by

$$(x_1^*, x_2^*) = \left(1 - \kappa(\alpha+\beta), \frac{\alpha+\gamma}{\rho}\right).$$

The Jacobian matrix, obtained using (9) and (10) and evaluated at the non-trivial fixed point, has purely imaginary eigenvalues. All initial points (except the fixed point) are located in closed orbits and that the model generates perpetual oscillations in capitalist economies, as shown in Figs. 1 and 2. The figures are depicted under the assumption of $\alpha = 0.03, \beta = 0.03, \gamma = 0.87, \kappa = 5, \rho = 1, x_1(0) = 0.6, x_2(0) = 0.85$. The non-trivial fixed point in this case is $(x_1^*, x_2^*) = (0.7, 0.9)$.

<u>Figs. 1 and 2 here</u>

The biological competition between predators and prey described by Lotka (1925) and Volterra (1931) can be interpreted in economic terms. Assumptions 1) to7) imply that investment financed by profits creates employment and that the rate of change in profit share is a decreasing function of the employment rate. As the profit share increases, employment expands, and the employment rate goes up, increasing workers' bargaining power, which lowers the profit share and depressing the employment rate, which again causes workers' bargaining power to weaken and the profit share to move up, increasing the employment rate.

3.2. Introduction of the optimal control into the Goodwin model

We regard x_1 and x_2 as state variables and introduce the control variable u and the cost function into Goodwin's (1967) model.[5]

We assume that the government has a cost function to minimize. We assume that the cost function has a linear quadratic form, defined by the differences between the target and actual levels of distributive share, employment rate, and fiscal spending. The dynamically optimal control problem (**DOC1**) is given as

$$\mathbf{DOC1}\ \min G_T = \min \frac{1}{2}\int_0^T [a_1(x_1(t)-b_1)^2 + a_2(x_2(t)-b_2)^2 + a_3(u(t)-b_3)^2]dt\,,$$

$$\text{s.t.}\,\dot{x}_1(t) = x_1(t)[-(\alpha+\gamma)+\rho x_2(t)], \quad (11)$$

$$\dot{x}_2(t) = x_2(t)\left[\frac{1-x_1(t)}{\kappa} - (\alpha+\beta) + a_4 u(t)\right], \qquad (12)$$

$$u(t) \geqq 0,$$

where G_T is the government cost function of **DOC1** and $u(t)$ denotes fiscal spending per worker. We assume $a_4 > 0$, indicating that fiscal spending has a positive effect on the employment rate. Therefore, (12) follows the principle of Keynesian effective demand.[6] $b_1, b_2, b_3 \geqq 0$ are the government targets of the wage share, employment rate, and fiscal spending, and $a_1, a_2, a_3 \geqq 0$ are the weight coefficients to the wage share, employment rate, and fiscal spending, respectively.

The static optimal control problem (**SOC1**) is defined as

$$\mathbf{SOC1}\ \min G = \min \frac{1}{2}\,[a_1(x_1-b_1)^2 + a_2(x_2-b_2)^2 + a_3(u-b_3)^2],$$

$$\text{s.t.}\,\{x_1[-(\alpha+\gamma)+\rho x_2] = 0, x_2\left[\frac{1-x_1}{\kappa} - (\alpha+\beta) + a_4 u\right] = 0, u \geqq 0,$$

where G is the government cost function of **SOC1**.

The Hamiltonian function of **DOC1** is given as

$$\begin{aligned}\mathcal{H}\big(x_1(t), x_2(t), \lambda_1(t), \lambda_2(t), u(t)\big) = \lambda_0 \frac{1}{2}[a_1(x_1(t)-b_1)^2 + a_2(x_2(t)-b_2)^2 + a_3(u(t)-b_3)^2] \\ +\lambda_1(t)x_1(t)[-(\alpha+\gamma)+\rho x_2(t)] \\ +\lambda_2(t)x_2(t)\left[\frac{1-x_1(t)}{\kappa} - (\alpha+\beta) + a_4 u(t)\right]. \quad (13)\end{aligned}$$

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[5] See the Appendix for the mathematical formulations in detail.

[6] The Rössler's dynamic model in Goodwin (1990a, 1990b, 1993) has a similar structure as Goodwin (1967) in that the rate of change of employment declines if the wage share increases and the rate of change in wage share increases if the employment rate increases.

Without loss of generality, we suppose $\lambda_0 = -1$.

The Lagrange function is given as

$$\mathcal{L}(x_1, x_2, \lambda_1, \lambda_2, u) = \lambda_0 \frac{1}{2}[a_1(x_1 - b_1)^2 + a_2(x_2 - b_2)^2 + a_3(u - b_3)^2]$$
$$+\lambda_1 x_1[-(\alpha+\gamma) + \rho x_2] + \lambda_2 x_2 \left[\frac{1 - x_1(t)}{\kappa} - (\alpha+\beta) + a_4 u\right].$$

The solution to **SOC1** is the turnpike.

3.3. The numerical analysis

We conduct a numerical analysis of the model in Subsection 3.2. We assume the parameters and initial values of the state variables are

$$\alpha = 0.03, \beta = 0.03, \gamma = 0.87, \kappa = 3.5, \rho = 1, a_1 = a_2 = a_3 = 1, a_4 = 0.01,$$
$$b_1 = 0.7, b_2 = 0.8, b_3 = 0.2, x_1(0) = 0.6, x_2(0) = 0.85.$$

Furthermore, we assume that the final period is $T = 100$ and the terminal position is free, which means $\boldsymbol{\lambda}_T(100) = \mathbf{0}$.

The Hessian matrix derived from (13) under the above parametric setting is given by

$$D^2\mathcal{H} = \begin{pmatrix} -1 & \lambda_1 - 0.286\lambda_2 & -0.9 + x_2 & -0.286x_2 & 0 \\ \lambda_1 - 0.286\lambda_2 & -1 & x_1 & -0.06 + 0.01u + 0.286(1 - x_1) & 0.01\lambda_2 \\ -0.9 + x_2 & x_1 & 0 & 0 & 0 \\ -0.286x_2 & -0.06 + 0.01u + 0.286(1 - x_1) & 0 & 0 & 0.01x_2 \\ 0 & 0.01\lambda_2 & 0 & 0.01x_2 & -1 \end{pmatrix},$$

from which we obtain $\boldsymbol{\mathcal{H}}_{xx} = \begin{pmatrix} -1 & \lambda_1 - 0.286\lambda_2 \\ \lambda_1 - 0.286\lambda_2 & -1 \end{pmatrix}$, $\boldsymbol{\mathcal{H}}_{xu} = \begin{pmatrix} 0 \\ 0.01\lambda_2 \end{pmatrix}$, $\boldsymbol{\mathcal{H}}_{uu} = -1$, $\boldsymbol{\mathcal{H}}_{\lambda x} = \begin{pmatrix} -0.9 + x_2 & x_1 \\ -0.286x_2 & -0.06 + 0.01u + 0.286(1 - x_1) \end{pmatrix}$, $\boldsymbol{\mathcal{H}}_{\lambda u} = \begin{pmatrix} 0 \\ 0.01x_2 \end{pmatrix}$. Under the parameters, the turnpike is given by $(\bar{x}_1, \bar{x}_2, \bar{u}, \bar{\lambda}_1, \bar{\lambda}_2) = (0.797, 0.9, 0.197, 0.111, -0.377)$.

We confirm that our model, with the given initial values and parameters, satisfies the conditions of Theorem 2. Using these parameters, we obtain

- $\boldsymbol{\mathcal{H}}_{uu} = -1$ satisfies the symmetric negative definite.
- $\boldsymbol{W} = \begin{pmatrix} 1 & -0.219 \\ -0.219 & 1 \end{pmatrix}$, whose eigenvalues are 1.22 and 0.781, is a symmetric positive definite matrix.
- $\boldsymbol{A} = \begin{pmatrix} 0 & 0.797 \\ -0.257 & 0.0201 \end{pmatrix}$, $\boldsymbol{B} = \begin{pmatrix} 0 \\ 0.009 \end{pmatrix}$. Thus, $\boldsymbol{AB} = \begin{pmatrix} 0.00171 \\ 0.000181 \end{pmatrix}$ is obtained. Therefore, $\mathrm{Rank}[\boldsymbol{B}, \boldsymbol{AB}] = \mathrm{Rank}\left[\begin{pmatrix} 0 & 0.00171 \\ 0.009 & 0.000181 \end{pmatrix}\right] = 2$, which is equal to the dimension of the state variable $\mathbb{R}^2$.
- $x_1(0) = 0.6$ and $\boldsymbol{x_2}(0) = 0.85$ are assumed. Then,

$$\bar{D} = \|(\bar{x}_1 - 0.6, \bar{x}_2 - 0.85)\| + \|(\bar{\lambda}_1, \bar{\lambda}_2)\| = 0.596.$$

Thus, we confirm the existence of a small $\varepsilon > 0$.

From the above results, the stabilizing solution of (A10), shown in the Appendix, is given by $\boldsymbol{P}_- = \begin{pmatrix} -188.37 & -1.94 \\ -1.94 & -583.11 \end{pmatrix}$, with eigenvalues of -583 and -188. $\boldsymbol{P}_-$ is a symmetric negative definite matrix. We obtain

$$\eta \equiv -\max\{\mathrm{Re}(\lambda_i) | \lambda_i \in \mathrm{spec}(\boldsymbol{A} + \boldsymbol{B}\boldsymbol{\mathcal{H}}_{uu}^{-1}\boldsymbol{B}^\top\boldsymbol{P}_-)\} = 0.0136.$$

As argued in the Appendix, C linearly depends on $\bar{D}$ and $e^{-0.0136\times 100} \cong 0$. Thus, $C = \mathcal{O}(1)$. Therefore, our analysis satisfies the assumptions in Theorem 2.

To demonstrate the turnpike property, we use E-view to compute the optimal solution by setting the iteration to 5000 with an error tolerance $= 10^{-8}$. The solution is obtained using the Broyden algorithm, and the simulation type is a deterministic and dynamic solution. Figs. 3, 4, and 5 show the evolution of the wage share, employment rate, and control variable, respectively. They show that x_1, x_2, and u remain close to the turnpike for considerable periods, although they oscillate in the early periods. The turnpike property holds in this model, in contrast to the fact that the Lotka-Volterra competition without optimal control shows a perpetual cycle, as shown in Figs. 1 and 2.

<u>*Figs. 3, 4, and 5 here.*</u>

It can be argued that the assumed parameters and initial values are unrealistic. Our purpose is to investigate the applicability of the turnpike property in nonlinear optimal control. The current stage is not where we demonstrate whether models applying the turnpike property can fit the actual data well.

4. Implications of the Model and Numerical Analysis

Goodwin (1967) has been considered as explaining the perpetual growth cycle that accompanies class struggle. Therefore, it has been often used to assert the instability of capitalist economies. When the model is disaggregated, structural economic dynamics occur, and each sector has a cycle depending on its own labor productivity growth rate and capital-output ratio. Goodwin (1986) argued that the oscillations generated by the growth cycle are "swinging along the turnpike" and the turnpike is unstable. Moreover, Goodwin (1990a, 1990b, 1993) considered the effects of government intervention on economic fluctuations by applying Rössler's dynamic model to economics, and showed that government intervention reduces the fluctuations.

The turnpike property develops this argument. We demonstrate that the optimal path of the model generalizing Goodwin (1967) remains close to the turnpike for considerable periods of time owing to the government's optimal behavior. The government has a stabilizing effect on structural change and growth cycle, at least in the medium term. x_1 and x_2 in our model correspond to the number of predators and prey, respectively, in the Lotka-Volterra competition and have the same competitive relationship as in Goodwin (1967). (12) is equivalent to assuming that the prey is released from the system.[7] Figs. 3, 4, and 5 show that the motion of u precedes that of x_2, followed by that of x_1. Because predators and prey compete in the same manner as in Goodwin (1967), x_1 and x_2 oscillate. However, the movement of the fiscal spending optimally controls the motions in x_1 and x_2 to minimize the cost, and eventually, $x_1, x_2,$ and u stay near the turnpike for considerable periods after a few oscillations.

Goodwin (1986) asserted that the 1967 model has an *unstable* turnpike. Although the (nontrivial) fixed point of this model indicates the turnpike, the trajectory obtained here oscillates around the turnpike. This implies that no economic system can get on the turnpike. By contrast, our model demonstrates that Goodwin can meet Neumann on the turnpike if we discover the relationship between the state and control variables, as given in (11) and (12), and the government appropriately sets the cost function and minimizes it.

As argued in Section 1, Goodwin (1990a,1990b, 1993) concluded that government behavior can reduce the amplitude of fluctuations in economic activity. This is a similar conclusion to ours. The difference between Goodwin's and our approach is, first, that Goodwin's result crucially depends on the specific system of differential equations, called the Rössler dynamic model, whereas our approach holds in broader classes of nonlinear economic models as long as the assumptions in Theorem 2 are satisfied. Second, our approach uses the optimal method, whereas Goodwin's approach does not. From the contemporary viewpoint of economic theory, our approach seems adequate because the government's objectives are clarified.

The government's stabilizing effect on economic growth and cycles is associated with Pasinetti's emphasis on the importance of *institutions* for stable economic growth (Pasinetti, 1993, 2007). Pasinetti (1993, pp. 117–118) argued that every society faces an *institutional problem*, which is searching for organizational devices to keep the economic system in equilibrium. He asserted that the solution to the *institutional problem* does not need to be unique. As Cardinale (2024) emphasized, a system actor must exist to design institutions. We assume that it is the government, and the government solves the optimal control problem, the objective function of which is the cost function to minimize. The government selects the parameters and forms the cost function. We can suppose that the choice of parameters and form set by the government reflects social value judgements, as in the case of the Bergson–Samuelson social welfare function (Pattanaik, 2008). Moreover, the choice of parameters and form affects the turnpike $(\bar{\boldsymbol{x}}, \bar{\boldsymbol{\lambda}}, \bar{\boldsymbol{u}})$, since the turnpike, which is nothing but the solution to **SOC**, depends on the parameters and form of the cost function. The turnpike in our model is efficient, at least in the sense that all constraints are binding, and is the ideal trajectory in this sense. Therefore, in our model, the government has the potential to choose an ideal trajectory and control the economic system such that the dynamically optimal path converges to it. We can consider the cost function as a measure for solving the *institutional problem*; the choice of the cost function parameter is not necessarily unique, which means the solution to the institutional problem may not be unique as well.

However, our model has certain limitations. First, we do not optimize production and consumption because our analysis follows Goodwin (1967) and assumes that capitalists invest all the profits they earn and workers entirely spend their wages on consumption. Furthermore, we do not specify the resources for fiscal spending. However,

[7] In contrast to this study, Ibañez (2017) showed that the turnpike property, defined by (5), holds in the Lotka-Volterra model for "hunting" predator and prey as the control variable.

we demonstrate that the turnpike property in nonlinear optimal control has high applicability in economics. In particular, the application of the property is promising in analytical fields rarely investigated because the models become very complicated owing to nonlinearity. For example, the turnpike property in nonlinear optimal control theory would be helpful when focusing on the stabilizing policy in the model with increasing returns to scale. As argued in Subsection 2.1, we do not need to solve the dynamically optimal control problem of such a complicated model as long as we can confirm that the assumptions in Theorem 2 are satisfied.

5. Concluding Remarks

We have examined the applicability of the turnpike property in nonlinear optimal control in economics. Although Goodwin's (1967) model exhibits a perpetual growth cycle, we demonstrate, using nonlinear optimal control theory, that the turnpike theorem holds. The results shed light on a new relationship between the turnpike and Goodwin (1967). Unlike in Goodwin (1986), the turnpike in Goodwin (1967) can be stabilized by assuming optimal government control of the economic system. Furthermore, the stabilizing effect of government activity in our model works better than in Goodwin (1990a, 1990b, 1993), as shown in Section 4.

The turnpike theorem in the nonlinear optimal control theory holds if the assumptions in Theorem 2 are satisfied. The broader classes of realistic economic models, including models with increasing returns to scale, are likely to meet these assumptions, especially when the linear quadratic cost function is used. This implies that the turnpike theorem can be applied to a broader class of economic models than assumed so far.

If the optimizing actor is assumed to be the government, we can consider how to stabilize an economic system that is intrinsically unstable owing to nonlinearity and structural economic dynamics. Goodwin and Pasinetti attempted to address this problem. Pasinetti's (2007) methodology is associated with the *institutional problem*. The choice of the parameters and form of the cost function affects the turnpike, which can be considered an ideal trajectory, at least in terms of efficiency. Our approach implies that the ideal trajectory may be affected by choosing the parameters and form of the cost function to be minimized by the government. This paves the way for developing arguments on economic policies to stabilize an intrinsically unstable economic system.

While our model has limitations, the turnpike property in nonlinear optimal control theory is a promising approach for examining the stability of an economic system. Such an economic system, for example, includes a multisectoral (multi-commodity) and nonlinear model. Although the economic system is realistic, it has rarely been analyzed because of the intractability of the model. However, by applying the turnpike property to the nonlinear optimal control theory, we can consider how to stabilize such an economic system.

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Mathematical Appendix

First, we argue for the turnpike property in the typical optimal control model, following Trélat and Zuazua (2015), and then deal with general cases.

The typical optimal control problem assumes that the cost function is linearly quadratic, and the state equation is given by (6). In this case, the Hamiltonian function of **DOC** is given by

$$\mathcal{H} = \lambda^0 \frac{1}{2}[(\boldsymbol{x}(t) - \boldsymbol{x}^a)^\top \boldsymbol{Q}(\boldsymbol{x}(t) - \boldsymbol{x}^a) + (\boldsymbol{u}(t) - \boldsymbol{u}^a)^\top \boldsymbol{U}(\boldsymbol{u}(t) - \boldsymbol{u}^a)] \\ + \boldsymbol{\lambda}^\top(t)\big(\boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}\boldsymbol{u}(t)\big),$$

where symmetric positive definite $\boldsymbol{Q} \in \mathbb{R}^{n\times n}$ and $\boldsymbol{U} \in \mathbb{R}^{m\times m}$ are the weight matrices to the state s and control variables, respectively, and $\boldsymbol{x}^a \in \mathbb{R}^n$ and $\boldsymbol{u}^a \in \mathbb{R}^m$ are arbitrary vectors.

Following the Pontryagin's maximum principle, the solution to **DOC** satisfies

$$\begin{cases} \dot{\boldsymbol{x}}_T(t) = \boldsymbol{A}\boldsymbol{x}_T(t) + \boldsymbol{B}\boldsymbol{U}^{-1}\boldsymbol{B}^\top\boldsymbol{\lambda}_T(t) + \boldsymbol{B}\boldsymbol{x}^a, & \text{(A1)} \\ \dot{\boldsymbol{\lambda}}_T(t) = \boldsymbol{Q}^\top\boldsymbol{x}_T(t) - \boldsymbol{A}^\top\boldsymbol{\lambda}_T(t) - \boldsymbol{Q}\boldsymbol{x}^a, & \text{(A2)} \\ \boldsymbol{u}_T(t) = \boldsymbol{u}^a + \boldsymbol{U}^{-1}\boldsymbol{B}^\top\boldsymbol{\lambda}_T(t)\,. & \text{(A3)} \end{cases}$$

The Lagrangian function of **SOC** is given by

$$\mathcal{L} = \lambda^0 \frac{1}{2}[(\boldsymbol{x} - \boldsymbol{x}^a)^\top \boldsymbol{Q}(\boldsymbol{x} - \boldsymbol{x}^a) + (\boldsymbol{u} - \boldsymbol{u}^a)^\top \boldsymbol{U}(\boldsymbol{u} - \boldsymbol{u}^a)] + \boldsymbol{\lambda}^\top(\boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}\boldsymbol{u}).$$

The solution to **SOC** satisfies

$$\begin{cases} \boldsymbol{A}\overline{\boldsymbol{x}} + \boldsymbol{B}\boldsymbol{U}^{-1}\boldsymbol{B}^\top\overline{\boldsymbol{\lambda}} = -\boldsymbol{B}\boldsymbol{x}^a, & \text{(A4)} \\ \boldsymbol{Q}\overline{\boldsymbol{x}} - \boldsymbol{A}^\top\overline{\boldsymbol{\lambda}} = \boldsymbol{Q}\boldsymbol{x}^a, & \text{(A5)} \\ \overline{\boldsymbol{u}} = \boldsymbol{u}^a + \boldsymbol{U}^{-1}\boldsymbol{B}^\top\overline{\boldsymbol{\lambda}}\,. & \text{(A6)} \end{cases}$$

Here, $\lambda^0 = -1$ can be assumed without loss of generality. The economic meaning of equations (A1)–(A6) is well-established (e.g., Léonard and van Long, 1992).

(A4) and (A5) are rewritten in a matrix form:

$$\boldsymbol{H}\begin{pmatrix}\overline{\boldsymbol{x}} \\ \overline{\boldsymbol{\lambda}}\end{pmatrix} = \begin{pmatrix}-\boldsymbol{B}\boldsymbol{x}^a \\ \boldsymbol{Q}\boldsymbol{x}^a\end{pmatrix}, \quad \text{(A7)}$$

where $\boldsymbol{H} \equiv \begin{pmatrix}\boldsymbol{A} & \boldsymbol{B}\boldsymbol{U}^{-1}\boldsymbol{B}^\top \\ \boldsymbol{Q} & -\boldsymbol{A}^\top\end{pmatrix} \in \mathbb{R}^{2n\times 2n}$. $\boldsymbol{H}$ is the Hamiltonian matrix, because of $\boldsymbol{B}\boldsymbol{U}^{-1}\boldsymbol{B}^\top = (\boldsymbol{B}\boldsymbol{U}^{-1}\boldsymbol{B}^\top)^\top$ and $\boldsymbol{Q} = \boldsymbol{Q}^\top$. The Hamiltonian matrix contains complex conjugate eigenvalues that are symmetric with respect to the origin of the complex plane. In other words, if λ is an eigenvalue of $\boldsymbol{H}$, $-\lambda$ is also the eigenvalue.

Since the pair $(\boldsymbol{A}, \boldsymbol{B})$ is assumed to satisfy (7), $\mathrm{null}(\boldsymbol{A}^\top) \cap \mathrm{null}(\boldsymbol{B}^\top) = \{\boldsymbol{0}\}$ holds (Trélat and Zuazua, 2015, Lemma 1). Then, $\boldsymbol{H}$ is the non-singular matrix. Therefore, (A7) has a unique solution.

Letting $\delta\boldsymbol{x}(t) \equiv \boldsymbol{x}_T(t) - \overline{\boldsymbol{x}}$ and $\delta\boldsymbol{\lambda}(t) \equiv \boldsymbol{\lambda}_T(t) - \overline{\boldsymbol{\lambda}}$ $(\delta \in \mathbb{R})$, (A1) and (A2) yield

$$\begin{cases} \delta\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\delta\boldsymbol{x}(t) + \boldsymbol{B}\boldsymbol{U}^{-1}\boldsymbol{B}^{\mathsf{T}}\delta\boldsymbol{\lambda}(t), \\ \delta\dot{\boldsymbol{\lambda}}(t) = \boldsymbol{Q}\delta\boldsymbol{x}(t) - \boldsymbol{A}^{\mathsf{T}}\delta\boldsymbol{\lambda}(t), \end{cases}$$

where $\delta\boldsymbol{x}(0) \equiv \boldsymbol{x}_T(0) - \bar{\boldsymbol{x}}$ and $\delta\boldsymbol{\lambda}(0) \equiv \boldsymbol{\lambda}_T(0) - \bar{\boldsymbol{\lambda}}$. It can be rewritten as

$$\dot{\boldsymbol{Z}}(t) = \boldsymbol{H}\boldsymbol{Z}(t), \quad \text{(A8)}$$

where $\boldsymbol{Z}(t) \equiv \begin{pmatrix} \delta\boldsymbol{x}(t) \\ \delta\boldsymbol{\lambda}(t) \end{pmatrix}$. The system has $\boldsymbol{Z}(0) \equiv \begin{pmatrix} \boldsymbol{x}_0 - \bar{\boldsymbol{x}} \\ \delta\boldsymbol{\lambda}(0) \end{pmatrix}$ and $\boldsymbol{Z}(T) \equiv \begin{pmatrix} \boldsymbol{x}(T) - \bar{\boldsymbol{x}} \\ -\bar{\boldsymbol{\lambda}} \end{pmatrix}$, if $\boldsymbol{x}_T(0) = \boldsymbol{x}_0$ and the terminal position is left free (i.e.,$\boldsymbol{\lambda}_T(T) = 0$).

Let $\boldsymbol{\lambda}(t) = \boldsymbol{P}(t)\boldsymbol{x}(t)$, where $\boldsymbol{P}(t) \in \mathbb{R}^{n\times n}$. Using the similar procedure in Intriligator (2002, p. 361) for (A8), we obtain the matrix Riccati differential equation:

$$\dot{\boldsymbol{P}}(t) + \boldsymbol{P}(t)\boldsymbol{A} + \boldsymbol{A}^{\mathsf{T}}\boldsymbol{P}(t) + \boldsymbol{P}(t)\boldsymbol{B}\boldsymbol{U}^{-1}\boldsymbol{B}^{\mathsf{T}}\boldsymbol{P}(t) - \boldsymbol{Q} = \boldsymbol{0}, \qquad \text{(A9)}$$

the unknowns of which are the elements of $\boldsymbol{P}$. The steady state ($\dot{\boldsymbol{P}} = \boldsymbol{0}$) is given by:

$$\boldsymbol{P}\boldsymbol{A} + \boldsymbol{A}^{\mathsf{T}}\boldsymbol{P} + \boldsymbol{P}\boldsymbol{B}\boldsymbol{U}^{-1}\boldsymbol{B}^{\mathsf{T}}\boldsymbol{P} - \boldsymbol{Q} = \boldsymbol{0}, \quad \text{(A10)}$$

which is called the algebraic Riccati equation.[8]

(A10) provides important information regarding the solution to (A8). Letting $\boldsymbol{X}, \boldsymbol{Y} \in \mathbb{R}^{n\times n}$, we can formulate the following equations based on (A8).

$$\boldsymbol{H}\begin{pmatrix} \boldsymbol{Y} \\ \boldsymbol{X} \end{pmatrix} = \begin{pmatrix} \boldsymbol{Y} \\ \boldsymbol{X} \end{pmatrix}\boldsymbol{\Lambda}, \qquad \text{(A11)}$$

where $\boldsymbol{\Lambda} \in \mathbb{R}^{n\times n}$ is the diagonal matrix, whose diagonals are the eigenvalues with the positive real parts. Thus, the i th column vector $\begin{pmatrix} \boldsymbol{y}_i \\ \boldsymbol{x}_i \end{pmatrix}$ is the eigenvector of $\boldsymbol{H}$ corresponding to the ith diagonal element λ_i (the eigenvalues). (A11) implies that

$$\begin{cases} \boldsymbol{A}\boldsymbol{Y} + \boldsymbol{B}\boldsymbol{U}^{-1}\boldsymbol{B}^{\mathsf{T}}\boldsymbol{X} = \boldsymbol{Y}\boldsymbol{\Lambda}, \\ \boldsymbol{Q}\boldsymbol{Y} - \boldsymbol{A}^{\mathsf{T}}\boldsymbol{X} = \boldsymbol{X}\boldsymbol{\Lambda}. \end{cases} \quad \text{(A12)}$$

Assume that $\boldsymbol{Y}$ is a non-singular matrix. By transforming (A12), we obtain

$$(\boldsymbol{X}\boldsymbol{Y}^{-1})\boldsymbol{A} + \boldsymbol{A}^{\mathsf{T}}(\boldsymbol{X}\boldsymbol{Y}^{-1}) + (\boldsymbol{X}\boldsymbol{Y}^{-1})\,\boldsymbol{B}\boldsymbol{U}^{-1}\boldsymbol{B}^{\mathsf{T}}(\boldsymbol{X}\boldsymbol{Y}^{-1}) - \boldsymbol{Q} = \boldsymbol{0}.$$

Putting $\boldsymbol{P} = \boldsymbol{X}\boldsymbol{Y}^{-1}$ yields (A10). Therefore, the solution to (A10) is obtained by combining the eigenvectors of $\boldsymbol{H}$.

Using (A3) and $\boldsymbol{\lambda}(t) = \boldsymbol{P}\boldsymbol{x}(t)$, the state equation is rewritten as

$$\begin{aligned} \dot{\boldsymbol{x}}(t) &= \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}\big(\boldsymbol{u}^a + \boldsymbol{U}^{-1}\boldsymbol{B}^{\mathsf{T}}\boldsymbol{\lambda}(t)\big), \\ &= (\boldsymbol{A} + \boldsymbol{B}\boldsymbol{K})\boldsymbol{x}(t) + \boldsymbol{B}\boldsymbol{u}^a, \end{aligned}$$

[8] The algebraic Riccati equation is used in economics when the dynamic programming problems with the linear quadratic value function are addressed (see Sargent, 1987).

where $\boldsymbol{K} \equiv \boldsymbol{U}^{-1}\boldsymbol{B}^{\top}\boldsymbol{P} \in \mathbb{R}^{m\times n}$.
Note that $\boldsymbol{K}$ depends on the solution of (A10).

Definition A.1 (Trélat, 2024, p. 64): *The system of (6) is said to be feedback stabilizable if there is a matrix* $\boldsymbol{K}$ *such that the closed-loop system with the feedback* $\boldsymbol{u}(t) = \boldsymbol{K}\boldsymbol{x}(t)$*,*

$$\dot{\boldsymbol{x}}(t) = (\boldsymbol{A} + \boldsymbol{B}\boldsymbol{K})\boldsymbol{x}(t) \text{ (A13)}$$

is asymptotically stable.

Remark A1. Definition A.1 implies that $\boldsymbol{A} + \boldsymbol{B}\boldsymbol{K}$ is Hurwitz.

Definition A2 (Molinari, 1973): *The stabilizing solution is the solution of (A10) given by a symmetric matrix, which satisfies*

$$\mathrm{Re}[\lambda(\boldsymbol{A} + \boldsymbol{B}\boldsymbol{K})] < 0,$$

where $\mathrm{Re}[\lambda(\boldsymbol{A} + \boldsymbol{B}\boldsymbol{K})]$ *denotes the real part of the eigenvalues of matrix* $\boldsymbol{A} + \boldsymbol{B}\boldsymbol{K}$*.*

Since the pair $(\boldsymbol{A}, \boldsymbol{B})$ is assumed to be controllable, there are no eigenvalues of $\boldsymbol{H}$ that has the property of $\mathrm{Re}[\lambda(\boldsymbol{H})] = 0$ (Kučera, 1972). Therefore, $\boldsymbol{H}$ has no purely imaginary eigenvalues.

Theorem A.1: *If the pair* $(\boldsymbol{A}, \boldsymbol{B})$ *is controllable, there exists at most one real symmetric solution of (A10) having the property that* $\mathrm{Re}[\lambda(\boldsymbol{A} + \boldsymbol{B}\boldsymbol{U}^{-1}\boldsymbol{B}^{\top}\boldsymbol{P})] < 0 (> 0)$*.*

Proof: See Willems (1971). ■

Theorem A.2: *Let* $\boldsymbol{P}_+$ *and* $\boldsymbol{P}_-$ *be real symmetric solution of (A10). If the pair* $(\boldsymbol{A}, \boldsymbol{B})$ *is controllable, (A10) has exactly one real symmetric negative definite solution (*$\boldsymbol{P}_-$*) having the property* $\mathrm{Re}[\lambda(\boldsymbol{A} + \boldsymbol{B}\boldsymbol{U}^{-1}\boldsymbol{B}^{\top}\boldsymbol{P}_-)] < 0$ *and has exactly one real symmetric positive definite solution (*$\boldsymbol{P}_+$*) having the property* $\mathrm{Re}[\lambda(\boldsymbol{A} + \boldsymbol{B}\boldsymbol{U}^{-1}\boldsymbol{B}^{\top}\boldsymbol{P}_+)] > 0$*. Letting* $\boldsymbol{P}$ *be other symmetric solutions of (A10),* $\boldsymbol{P}_- \leq \boldsymbol{P} \leq \boldsymbol{P}_+$*.*

Proof: See Willems (1971). ■

Theorem A.2 implies that $\boldsymbol{A} + \boldsymbol{B}\boldsymbol{U}^{-1}\boldsymbol{B}^{\top}\boldsymbol{P}_-$ is the Hurwitz matrix.

Let us transform the variables as follows.

$$\boldsymbol{Z}(t) = \begin{pmatrix} \boldsymbol{I}_n & \boldsymbol{I}_n \\ \boldsymbol{P}_- & \boldsymbol{P}_+ \end{pmatrix} \boldsymbol{Q}(t), \qquad \text{(A14)}$$

where $\boldsymbol{Q}(t) \equiv \begin{pmatrix} \boldsymbol{v}(t) \\ \boldsymbol{\omega}(t) \end{pmatrix} \in \mathbb{R}^{2n}$, and $\boldsymbol{I}_n \in \mathbb{R}^{n\times n}$ is the identity matrix. Using (A14), as Anderson and Kokotovic (1987) showed, (A8) can be transformed as

$$\dot{\boldsymbol{Q}}(t) = \begin{pmatrix} \boldsymbol{A} + \boldsymbol{B}\boldsymbol{U}^{-1}\boldsymbol{B}^{\top}\boldsymbol{P}_- & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{A} + \boldsymbol{B}\boldsymbol{U}^{-1}\boldsymbol{B}^{\top}\boldsymbol{P}_+ \end{pmatrix} \boldsymbol{Q}(t). \text{ (A15)}$$

from which, we have

$$\begin{cases} \dot{\boldsymbol{v}}(t) = (\boldsymbol{A} + \boldsymbol{B}\boldsymbol{U}^{-1}\boldsymbol{B}^{\intercal}\boldsymbol{P}_{-})\boldsymbol{v}(t), \\ \dot{\boldsymbol{\omega}}(t) = (\boldsymbol{A} + \boldsymbol{B}\boldsymbol{U}^{-1}\boldsymbol{B}^{\intercal}\boldsymbol{P}_{+})\boldsymbol{\omega}(t). \end{cases}$$

As already shown, $\mathrm{Re}[\lambda(\boldsymbol{A} + \boldsymbol{B}\boldsymbol{U}^{-1}\boldsymbol{B}^{\intercal}\boldsymbol{P}_{-})] < 0$ and $\mathrm{Re}[\lambda(\boldsymbol{A} + \boldsymbol{B}\boldsymbol{U}^{-1}\boldsymbol{B}^{\intercal}\boldsymbol{P}_{+})] > 0$.[9] The system (A15) is purely hyperbolic, and thus $\boldsymbol{v}(t)$ is decaying and $\boldsymbol{\omega}(t)$ are expanding for $t \in [0, T]$:

$$\|\boldsymbol{v}(t)\| \leqq \|\boldsymbol{v}(0)\| e^{-\eta t} \text{ and } \|\boldsymbol{\omega}(t)\| \leqq \|\boldsymbol{\omega}(T)\| e^{-\eta(T-t)}, \text{(A16)}$$

where $\eta \equiv -\max_{i=1,\cdots,n}\{\mathrm{Re}(\lambda_i) | \lambda_i \in \mathrm{spec}(\boldsymbol{A} + \boldsymbol{B}\boldsymbol{U}^{-1}\boldsymbol{B}^{\intercal}\boldsymbol{P}_{-})\} > 0$. From (A16), $\boldsymbol{v}(t) \simeq \boldsymbol{0}$ and $\boldsymbol{\omega}(t) \simeq \boldsymbol{0}$ imply $\delta\boldsymbol{x}(t) \simeq \boldsymbol{0}$ and $\delta\boldsymbol{\lambda}(t) \simeq \boldsymbol{0}$, which also imply $\delta\boldsymbol{u}(t) \simeq \boldsymbol{0}$. Intuitively, $\big(\delta\boldsymbol{x}(t), \delta\boldsymbol{\lambda}(t), \delta\boldsymbol{u}(t)\big) \simeq \{\boldsymbol{0}\}$ implies that the optimal trajectory remains close to the steady state $\big(\bar{\boldsymbol{x}}, \bar{\boldsymbol{\lambda}}, \bar{\boldsymbol{u}}\big)$.

From (A14), $\delta\boldsymbol{x}(0) = \boldsymbol{v}(0) + \boldsymbol{\omega}(0)$. (A16) implies:

$$\|\boldsymbol{v}(0) - (\boldsymbol{x}(0) - \bar{\boldsymbol{x}})\| = \|\boldsymbol{\omega}(0)\| \leqq \|\boldsymbol{\omega}(T)\| e^{-\eta T}. \text{ (A17)}$$

Since we assume that $\boldsymbol{x}_T(T)$ is left free, $\boldsymbol{\lambda}_T(T) = \boldsymbol{0}$ holds. By (A14), therefore, $\delta\boldsymbol{\lambda}(T) = -\bar{\boldsymbol{\lambda}} = \boldsymbol{P}_{-}\boldsymbol{v}(T) + \boldsymbol{P}_{+}\boldsymbol{\omega}(T)$, which implies:

$$\left\|\boldsymbol{\omega}(T) + \boldsymbol{P}_{+}^{-1}\bar{\boldsymbol{\lambda}}\right\| \leqq \|\boldsymbol{P}_{+}^{-1}\boldsymbol{P}_{-}\| \boldsymbol{v}(0) e^{-\eta T}. \text{ (A18)}$$

Multiplying (A17) by $e^{-\eta T}$ yields

$$\|\boldsymbol{v}(0)\| e^{-\eta T} \leqq \|\boldsymbol{x}(0) - \bar{\boldsymbol{x}}\| e^{-\eta T} + \|\boldsymbol{\omega}(T)\| e^{-2\eta T}.$$

Substituting this into (A18), we obtain

$$\boldsymbol{\omega}(T) = -\boldsymbol{P}_{+}^{-1}\bar{\boldsymbol{\lambda}} + \mathcal{O}(\|\boldsymbol{P}_{+}^{-1}\boldsymbol{P}_{-}\| \|\boldsymbol{x}(0) - \bar{\boldsymbol{x}}\| e^{-\eta T}), \quad \text{(A19)}$$

where $\mathcal{O}$ is the Landau symbol. By the same token, $\|\boldsymbol{v}(0) - (\boldsymbol{x}(0) - \bar{\boldsymbol{x}})\| \leqq \|\boldsymbol{P}_{+}^{-1}\boldsymbol{P}_{-}\| e^{-\eta T} + \|\boldsymbol{P}_{+}^{-1}\boldsymbol{P}_{-}\| \boldsymbol{v}(0) e^{-2\eta T}$ holds. Thus, we obtain

$$\boldsymbol{v}(0) = \boldsymbol{x}(0) - \bar{\boldsymbol{x}} + \mathcal{O}\big(\|\boldsymbol{P}_{+}^{-1}\boldsymbol{P}_{-}\| (e^{-\eta T})\big). \quad \text{(A20)}$$

(A19) and (A20) determine $\big(\boldsymbol{v}(0), \boldsymbol{\omega}(T)\big)$. From inequality (A16), we obtain

$$\|\boldsymbol{v}(t)\| \leqq \|\boldsymbol{x}(0) - \bar{\boldsymbol{x}}\| e^{-\eta T} + \mathcal{O}\big(\left\|\boldsymbol{P}_{+}^{-1}\bar{\boldsymbol{\lambda}}\right\| e^{-\eta(t+T)}\big), \qquad \text{(A21)}$$

$$\|\boldsymbol{\omega}(t)\| \leqq \left\|\boldsymbol{P}_{+}^{-1}\bar{\boldsymbol{\lambda}}\right\| e^{-\eta(T-t)} + \mathcal{O}\big(\|\boldsymbol{P}_{+}^{-1}\boldsymbol{P}_{-}\| \|\boldsymbol{x}(0) - \bar{\boldsymbol{x}}\| e^{-\eta(2T-t)}\big), \qquad \text{(A22)}$$

for $t \in [0, T]$. From (A14), (A21), and (A22), we obtain

[9] It implies that $\boldsymbol{P}_{-}$ is the stabilizing solution of (A10).

$$\|\delta\boldsymbol{x}(t)\| \leqq \|\boldsymbol{x}(0)-\overline{\boldsymbol{x}}\|e^{-\eta T} + \left\|\boldsymbol{P}_{+}^{-1}\overline{\boldsymbol{\lambda}}\right\|e^{-\eta(T-t)}$$
$$+\,\mathcal{O}\left(\left\|\boldsymbol{P}_{+}^{-1}\overline{\boldsymbol{\lambda}}\right\|e^{-\eta(t+T)} + \|\boldsymbol{P}_{+}^{-1}\boldsymbol{P}_{-}\|\|\boldsymbol{x}(0)-\overline{\boldsymbol{x}}\|e^{-\eta(2T-t)}\right), \text{(A23)}$$

$$\|\delta\boldsymbol{\lambda}(t)\| \leqq \|\boldsymbol{P}_{-}\|\|\boldsymbol{x}(0)-\overline{\boldsymbol{x}}\|e^{-\eta T} + \|\boldsymbol{P}_{+}\|\left\|\boldsymbol{P}_{+}^{-1}\overline{\boldsymbol{\lambda}}\right\|e^{-\eta(2T-t)}$$
$$+\mathcal{O}\left(\|\boldsymbol{P}_{-}\|\left\|\boldsymbol{P}_{+}^{-1}\overline{\boldsymbol{\lambda}}\right\|e^{-\eta(t+T)} + \|\boldsymbol{P}_{+}\|\|\boldsymbol{P}_{+}^{-1}\boldsymbol{P}_{-}\|\|\boldsymbol{x}(0)-\overline{\boldsymbol{x}}\|e^{-\eta(2T-t)}\right), \text{(A24)}$$

$$\|\delta\boldsymbol{u}_T(t)\| \leqq \|\boldsymbol{U}^{-1}\boldsymbol{B}^{\mathsf{T}}\|\|\delta\boldsymbol{\lambda}(t)\|. \text{ (A25)}$$

(A23)–(A25) show that $\|\delta\boldsymbol{x}(t)\|$, $\|\delta\boldsymbol{\lambda}(t)\|$ and $\|\delta\boldsymbol{u}_T(t)\|$ crucially depend on $\|\boldsymbol{x}(0)-\overline{\boldsymbol{x}}\|$ as well, which implies that C defined in Definition 1 also depends on $\|\boldsymbol{x}(0)-\overline{\boldsymbol{x}}\|$. As we argue in Remark 2, $\|\boldsymbol{x}(0)-\overline{\boldsymbol{x}}\|$ is closely related to the satisfaction of (8). We can state that the smaller $\overline{D}$ corresponds to a smaller C. Thus, (A23)–(A25) imply (5).

In general cases where the cost function is not linearly quadratic and the state equation does not have the form of (6), the Hamiltonian matrix, which is derived in a similar way when we obtain matrix $\boldsymbol{H}$, is given by

$$\boldsymbol{M} = \begin{pmatrix} \boldsymbol{\mathcal{H}}_{\lambda x} - \boldsymbol{\mathcal{H}}_{\lambda u}\boldsymbol{\mathcal{H}}_{uu}^{-1}\boldsymbol{\mathcal{H}}\boldsymbol{H}_{ux} & -\boldsymbol{\mathcal{H}}_{\lambda u}\boldsymbol{\mathcal{H}}_{uu}^{-1}\boldsymbol{\mathcal{H}}_{u\lambda} \\ -\boldsymbol{\mathcal{H}}_{xx} + \boldsymbol{\mathcal{H}}_{xu}\boldsymbol{\mathcal{H}}_{uu}^{-1}\boldsymbol{\mathcal{H}}_{ux} & -(\boldsymbol{\mathcal{H}}_{x\lambda} - \boldsymbol{\mathcal{H}}_{xu}\boldsymbol{\mathcal{H}}_{uu}^{-1}\boldsymbol{\mathcal{H}}_{u\lambda}) \end{pmatrix} = \begin{pmatrix} \boldsymbol{A} & -\boldsymbol{B}\boldsymbol{\mathcal{H}}_{uu}^{-1}\boldsymbol{B}^{\mathsf{T}} \\ \boldsymbol{W} & -\boldsymbol{A}^{\mathsf{T}} \end{pmatrix}.$$

According to Trélat and Zuazua (2015), the optimal control theory usually assumes that the symmetric matrix $\boldsymbol{\mathcal{H}}_{uu}$ is negative definite, which is called a *strong Legendre-Clesch condition* (Bryson and Ho, 1975, p. 135). By adding the assumption that symmetric matrix $\boldsymbol{W}$ is positive definite, the assumptions for the validity of Theorem 2 are made in such a way that $\boldsymbol{M}$ has the same characteristics as $\boldsymbol{H}$ in the first order, with remainder terms in $\mathcal{O}(\cdot)$. This implicitly means that $\|\delta\boldsymbol{x}(t)\| + \|\delta\boldsymbol{\lambda}(t)\| + \|\delta\boldsymbol{u}(t)\|$ remains small.[10]

Thus, the similar logical sequence to the case with the linear quadratic cost function and (6) holds in general cases, while the validity of (2) and (4) is more technical (Trélat and Zuazua, 2015, pp. 109–112).

[10] Submatrix $\boldsymbol{A}$ in matrix $\boldsymbol{M}$ is not the exact $\boldsymbol{A}$ in $\boldsymbol{H}$ (note that $\boldsymbol{\mathcal{H}}_{ux} = \boldsymbol{0}$ in the linear quadratic case). Thus, $\boldsymbol{A}$ in $\boldsymbol{M}$ is a deformation of $\boldsymbol{\mathcal{H}}_{\lambda x}$ with terms of the second order (Trélat and Zuazua, 2015).

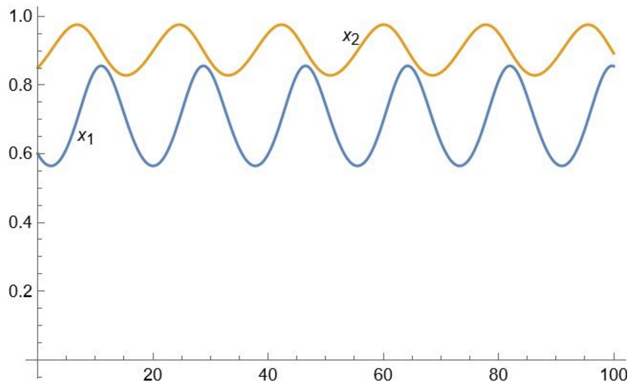

Figure 1: the dynamics of x_1, x_2

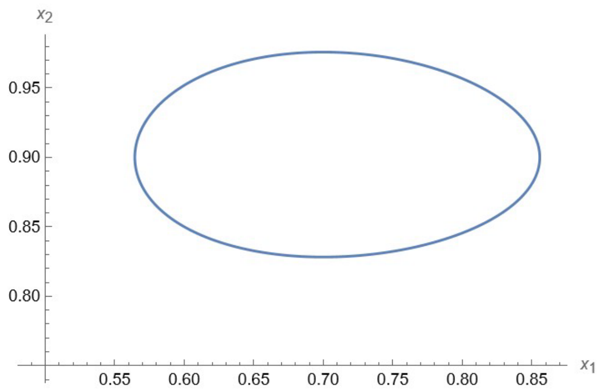

Figure 2: The phase diagram

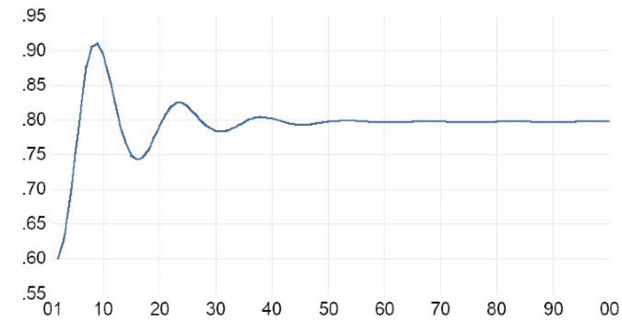

Figure 3: Wage share (x_1)

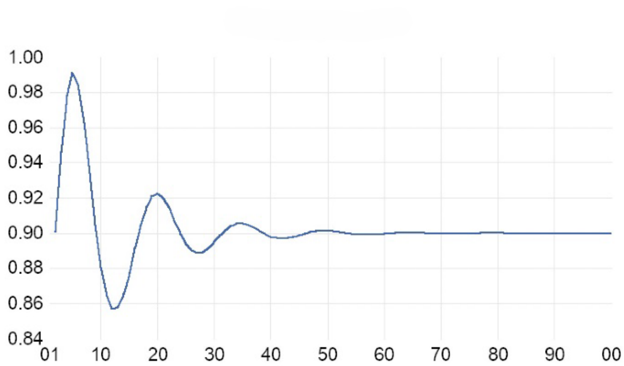

Figure 4: Employment rate (x_2)

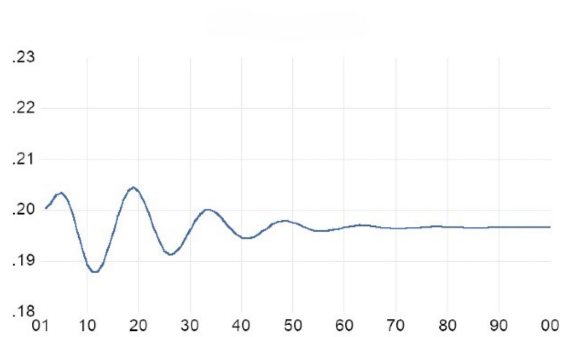

Figure 5: Fiscal spending (u)