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On the multiplicative law of subjective probability

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Abstract: This note elaborates on Luce and Narens' (1978) axiomatic derivation of subjective probability, which generically satisfies the multiplicative law by removing requirement for the derived subjective probability to be non-atomic. For the two original sample spaces, we add the sample space defined by the direct product of the original sample spaces and the sample space of the auxiliary experiment. As a main result, the necessary and sufficient conditions are provided for the likelihood relation on the events of the sample spaces to be represented by the subjective probability satisfying the law with respect to the direct product, allowing atoms in the original sample spaces.

Key words: subjective probability, multiplicative law, likelihood relation, auxiliary experiment

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1. Introduction

The subjective probability theory attempts to derive a probability measure as a (normalized) numerical indicator representing a decision-maker's likelihood relation on the sample space.[1] In particular, assuming that the sample space is given by the direct product of two sample spaces, Luce and Narens (1978) axiomatically derived the subjective probability measures, which generically satisfy the *multiplicative law* with respect to the direct product; however, the subjective probability measures are derived within the class of *non-atomic* measures. Because all probability measures in a finite sample space will never be non-atomic, the requirement for being a non-atomic measure is restrictive. This note attempts to elaborate on Luce and Narens' (1978, Theorem 5) axiomatic derivation, without requiring the derived subjective probability to be non-atomic.[2]

In the decision-maker's environment, we assume the presence of two original experiments (Experiments 1 and 2) initially. We then construct a subjctive probability measure on each of the sample spaces corresponding to the two experiments. To consider the multiplicative law between the two subjctive probability measures, we introduce the third experiment (Experiment 3) in which sample space is defined by the direct product of the two sample spaces. As the fourth experiment (Experiment 4), the auxiliary experiment by DeGroot (1970, Section 6.3) and French (1982, 2022) is introduced to determine the subjective probability of an event in the other sample spaces above based on the relative size of the equally likely event in the auxiliary experiment.

[1] For the survey articles, see Fishburn (1986, 1994).

[2] In Savage's scheme, where both the subjective probabilityand the expected utility are derived from the preference ordering, Grabisch, Monet and Vergopoulos (2023, Theorem 7) derive the subjective probability satisfying the multiplicative law in case of the non-atomic subjective probability.

First, we introduce French's (1982, 2022) five conditions for the likelihood relation on the events of the four sample spaces, assuming that the decision-maker's likelihood comparisons are possible for any pair of events, even if the events are selected from the different sample spaces. As a consequence of French (1982, 2022 Lemma 1), we show that the five conditions are necessary and sufficient for the likelihood relation to be represented by subjective probability, allowing atoms in the original sample spaces (Theorem 1).

Second, the domain of the likelihood relation is extended by including all of the conditional evens in the sample spaces. This extension is required for defining a condition called "Independence of irrelevant events" (IIE), which is a key condition for the main theorem stated below.[3] We assume that the decision-maker's likelihood comparisons are possible for any pair of conditional events, even if the conditioning events are different in the pair. Under the five conditions for Theorem 1, the assumption of three additional conditions is shown to be necessary and sufficient for the likelihood relation on the extended domain to be represented by the conditional probability, and that each of the three additional conditions is indispensable for the axiomatic derivation of the conditional probability (Theorem 2).

As the main result, under the eight conditions for Theorems 1 and 2, the assumption of two additional conditions, including the IIE condition on the likelihood relation, is is shown to be necessary and sufficient for the subjective probability to satisfy the multiplicative law, and that each of the two additional conditions is indispensable for the

[3] Based on the condition that $A \sim A|B$ in Luce and Narens (1978, Theorem 3), we define the IIE condition in terms of the likelihood relation. The IIE condition is closely related to the stochastic independence condition in the subjective expected utility theory such as Grabisch, Monet and Vergopoulos (2023, Axiom A7).

axiomatic derivation (Theorem 3). Hence, the subjective probabilities satisfying the multiplicative law are axiomatically derived from the full class of finite sample spaces, including the repeated coin-tossing experiments.

Within the class of coin-tossing experiments, we construct an example where Theorem 3 does not hold if the IIE condition is dropped in Theorem 3 to prove the indispensability of the IIE condition for Theorem 3. Specifically, in the case of the coin-tossings where the decision-maker will be informed of the outcome of the first tossing before the second tossing, the IIE condition requires that the likelihood of an outcome of the second tossing is invariant, whichever outcome occurs at the first tossing. If the initial belief (likelihood) before the first tossing is firm, then the belief may not be modified, and the IIE condition holds for such a decision-maker.

In the next section, we introduce the sample spaces and the likelihood relation on the events of the sample spaces.

2. Sample spaces and likelihood relation

Suppose that two original experiments exist, Experiments 1 and 2. For i = 1, 2, let S_i be the set of possible outcomes, called the *sample space* of Experiment i. Let $\mathfrak{B}_i$ be a σ-field of subsets of S_i, that is, $\mathfrak{B}_i$ is a set of subsets of S_i that is closed under complementation and σ-additivity. A set in $\mathfrak{B}_i$ is called an *event* in S_i. Specifically, S_i is called the *total event* of $\mathfrak{B}_i$. For the purpose of convenience, the empty set in $\mathfrak{B}_i$ is denoted by $\emptyset_i$. For a given pair of events A and B in $\mathfrak{B}_i$, a *conditional event* is denoted by an ordered pair A | B, where A | B represents an event A conditioned on an event B. The set of all conditional events on S_i is defined by $\Gamma_i = \{ A \mid B : A \in \mathfrak{B}_i \text{ and } B \in \mathfrak{B}_i. \}$.

To consider the independence between Experiments 1 and 2, we introduce the third experiment (Experiment 3) in which sample space S is defined by the *direct product*, S =

$S_1 \times S_2 = \{ (s_1, s_2) : s_1 \in S_1$ and $s_2 \in S_2 \}$, and denote $\mathfrak{B}_S = \mathfrak{B}_1 \times \mathfrak{B}_2$.[4] A set in $\mathfrak{B}_S$ is called an event in S. Specifically, S is called the total event of $\mathfrak{B}_S$, and the empty set in $\mathfrak{B}_S$ is denoted by $\emptyset_S$. For a given pair of events A and B in $\mathfrak{B}_S$, a conditional event is denoted by an ordered pair A|B, where A|B represents an event A conditioned on an event B. The set of all conditional events on S is defined by $\Gamma_S = \{ A|B : A \in \mathfrak{B}_S$ and $B \in \mathfrak{B}_S \}$.

The fourth experiment, Experiment 4, is introduced as French's auxiliary experiment in which sample space is the unit interval $T \equiv [0, 1]$. Let $\mathfrak{B}_T$ be the set of all Borel subsets in T. A set in $\mathfrak{B}_T$ is called an event in T. Specifically, T is called the total event of $\mathfrak{B}_T$, the empty set in $\mathfrak{B}_T$ is denoted by $\emptyset_T$, and the set of intervals in T is denoted by $\mathfrak{B}_T^*$. We assume that $\{ a \} \equiv [a, a] \in \mathfrak{B}_T^*$ for all $a \in T$, and we introduce the following two transformations for the events in $\mathfrak{B}_T^*$:

Translation to the right: Define A+c by $A+c = \{ x \in T : x = y + c$ for some $y \in A \}$ for all $A \in \mathfrak{B}_T^*$ with $\sup A < 1$ and all $c \in (0, 1 - \sup A]$.

Proportional shrink: Define c·A by $c \cdot A = \{ x \in T : x = c \cdot y$ for some $y \in A \}$ for all $A \in \mathfrak{B}_T^*$ and all $c \in (0, 1)$.

In the case of sample space T, a conditional event is denoted by an ordered pair A|B for A, $B \in \mathfrak{B}_T$. The set of all conditional events on T is defined by $\Gamma_T = \{ A|B : A \in \mathfrak{B}_T$ and $B \in \mathfrak{B}_T \}$.

A *likelihood relation* is a complete and transitive binary relation $\gtrsim$ on its domain $\mathcal{D} \subset \Gamma_1 \cup \Gamma_2 \cup \Gamma_S \cup \Gamma_T$. A likelihood relation is denoted by $(\mathcal{D}, \gtrsim)$. For A|B, C|D $\in \mathcal{D}$, the expression A|B $\gtrsim$ C|D means that the likelihood of A conditioned on B is greater than the likelihood of C conditioned on D. The symmetric and asymmetric parts of $\gtrsim$ are denoted by $\sim$ and $\succ$, respectively. We assume that

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[4] $\mathfrak{B}_S$ is the minimal σ-field in the class of all σ-fields of the subsets of $S_1 \times S_2$ containing all the product sets $A_1 \times A_2$, where $A_i \in \mathfrak{B}_i$ for $i = 1, 2$. See Itô (1984, Ch.2, Exercise 2.1(viii)) for the definition.

$\mathcal{D} = \{ A \mid B \in \Gamma_1 \cup \Gamma_2 \cup \Gamma_S \cup \Gamma_T : B \succ \emptyset_T \}$.

Moreover, to simplify the arguments, we introduce a definition and an assumption: let η be a function on $\mathfrak{B}_1 \cup \mathfrak{B}_2 \cup \mathfrak{B}_S \cup \mathfrak{B}_T$ defined by

$$\eta(A) = S_1 \text{ if } A \in \mathfrak{B}_1;$$

$$= S_2 \text{ if } A \in \mathfrak{B}_2;$$

$$= S \text{ if } A \in \mathfrak{B}_S;$$

$$= T \text{ if } A \in \mathfrak{B}_T.$$

Let A be an event in $\mathfrak{B}_1 \cup \mathfrak{B}_2 \cup \mathfrak{B}_S \cup \mathfrak{B}_T$. Because the conditional event $A \mid \eta(A)$ can be recognized as the event A itself, we assume that

$$A = A \mid \eta(A) \text{ for all } A \in \mathfrak{B}_1 \cup \mathfrak{B}_2 \cup \mathfrak{B}_S \cup \mathfrak{B}_T,$$

and we use the expression "$A \succsim B$" instead of "$A \mid \eta(A) \succsim B \mid \eta(B)$" when $A \mid \eta(A), B \mid \eta(B) \in \mathcal{D}$ in the remainder of this paper.[5]

3. French's theorem for the existence of subjective probability

To ensure the existence of the subjective probability, French (1982, 2022 Lemma 1) provides the necessary and sufficient conditions for the likelihood relation to be represented by the subjective probability. Concretely, we call the conditions axioms and we introduce French's axioms for the likelihood relation $(\mathcal{D}, \succsim)$ as follows:

A1(Total events and empty events): $S_1 \sim S_2 \sim S \sim T \succ \emptyset_1 \sim \emptyset_2 \sim \emptyset_S \sim \emptyset_T$ and $T \succsim A \succsim \emptyset_T$ for all $A \in \mathfrak{B}_1 \cup \mathfrak{B}_2 \cup \mathfrak{B}_S \cup \mathfrak{B}_T$.

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[5] The set $\Gamma_1 \cup \Gamma_2$ can be embedded into Γ_3 by the embedding rule that $A \times S_2 \mid S_1 \times S_2 = A \mid S_1$ for all $A \in \mathfrak{B}_1$ and $S_1 \times B \mid S_1 \times S_2 = B \mid S_2$ for all $B \in \mathfrak{B}_2$. Because the embedding rule requires that $S_1 = S_1 \mid S_1 = S_1 \times S_2 \mid S_1 \times S_2 = S_2 \mid S_2 = S_2$, we do not embed $\Gamma_1 \cup \Gamma_2$ into Γ_3.

Because $\eta(A) \succ \emptyset_T$ for all $A \in \mathfrak{B}_1 \cup \mathfrak{B}_2 \cup \mathfrak{B}_S \cup \mathfrak{B}_T$ under A1, we have the following lemma:

Lemma 1: $\mathfrak{B}_1 \cup \mathfrak{B}_2 \cup \mathfrak{B}_S \cup \mathfrak{B}_T \subset \mathcal{D}$, whenever the likelihood relation $(\mathcal{D}, \succsim)$ satisfies A1.

The next axiom is the additivity:

A2(Additivity): For any A, B, C, D $\in \mathfrak{B}_1 \cup \mathfrak{B}_2 \cup \mathfrak{B}_S \cup \mathfrak{B}_T$ such that $\eta(A) = \eta(B)$, $\eta(C) = \eta(D)$, $A \cap B = \emptyset_{\eta(A)}$ and $C \cap D = \emptyset_{\eta(C)}$, $(A \succsim C, B \succsim D) \Rightarrow A \cup B \succsim C \cup D$ holds, and $(A \succ C, B \succsim D) \Rightarrow A \cup B \succ C \cup D$ holds.

A3(Continuity): Let $\{ B_n \}$ be a sequence of events in $\mathfrak{B}_1 \cup \mathfrak{B}_2 \cup \mathfrak{B}_S \cup \mathfrak{B}_T$. If $B_{n+1} \subset B_n$ for all $n = 1, 2, \cdots$, and if there exists $A \in \mathfrak{B}_1 \cup \mathfrak{B}_2 \cup \mathfrak{B}_S \cup \mathfrak{B}_T$ such that $B_n \succsim A$ for all $n = 1, 2, \cdots$, then $\cap_n B_n \succsim A$, where $\cap_n B_n \equiv \cap_{n=1}^{\infty} B_n$.

A4(Positivity in $\mathfrak{B}_T^*$): $\sup A > \inf A \Leftrightarrow A \succ \emptyset_T$ for all $A \in \mathfrak{B}_T^*$.

A5(Invariance against translations to the right in $\mathfrak{B}_T^*$): $A \sim (A+c)$ for all $A \in \mathfrak{B}_T^*$ with $\sup A < 1$ and all $c \in (0, 1 - \sup A]$.

The three axioms, A1−A3 are assumed in French (1982, 2022), and this paper introduces A4 and A5, each of which is weaker than SP2 in French (1982) and PSW2 in French (2022).[6]

We have the following theorem:

Theorem 1 (French 1982 Theorem, French 2022 Lemma 1b): (**A**) If a likelihood relation $(\mathcal{D}, \succsim)$ satisfies the axioms A1−A5, then a real-valued function π uniquely exists on $\mathfrak{B}_1 \cup \mathfrak{B}_2 \cup \mathfrak{B}_S \cup \mathfrak{B}_T$ with the following five properties:

P1(Probability on $\mathfrak{B}_1$): The restriction of π on $\mathfrak{B}_1$ coincides with a probability measure on S_1.

P2(Probability on $\mathfrak{B}_2$): The restriction of π on $\mathfrak{B}_2$ coincides with a probability measure on S_2.

P3(Probability on $\mathfrak{B}_S$): The restriction of π on $\mathfrak{B}_S$ coincides with a probability measure on S.

P4(Uniform distribution on T): The restriction of π on $\mathfrak{B}_T$ coincides with the Lebesgue

[6] The two axioms, SP2 in French (1982) and PSW2 in French(2022) hold under A1-A5, as shown by Lemma 2(v) of this paper. Moreover, SP5 in French (1982) and PSW6 in French(2022) hold under A1-A5, as shown by Lemma 2(viii, ix).

measure μ on T.

P5(Preservation of ordering on events): $\pi(A) \geq \pi(B) \Leftrightarrow A \succsim B$ for all A, B ∈ $\mathfrak{B}_1 \cup \mathfrak{B}_2 \cup \mathfrak{B}_S \cup \mathfrak{B}_T$.

(**B**) If a real-valued function π* exists on $\mathfrak{B}_1 \cup \mathfrak{B}_2 \cup \mathfrak{B}_S \cup \mathfrak{B}_T$ having the properties P1−P4, then the binary relation ≿* defined on $\mathfrak{B}_1 \cup \mathfrak{B}_2 \cup \mathfrak{B}_S \cup \mathfrak{B}_T$ by $A \succsim^* B \Leftrightarrow \pi^*(A) \geq \pi^*(B)$ for all A, B ∈ $\mathfrak{B}_1 \cup \mathfrak{B}_2 \cup \mathfrak{B}_S \cup \mathfrak{B}_T$ satisfies A1−A5.

Theorem 1 implies that the five axioms A1−A5 are necessary and sufficient for the likelihood relation on the unconditional events to be represented by the probability measure. Although Theorem 1 has been shown by French (1982), the proof of Theorem 1 is provided in Section 6 for the completeness of this paper.

4. A theorem for the subjective conditional probability

For a given likelihood relation ($\mathcal{D}$, ≿) satisfying the axioms A1−A5, it holds by Theorem 1(A) that a real-valued function π exists on $\mathfrak{B}_1 \cup \mathfrak{B}_2 \cup \mathfrak{B}_S \cup \mathfrak{B}_T$ having the properties P1−P5. We call the function π the *probability function* of ($\mathcal{D}$, ≿). This section considers when the probability function also satisfies the following condition:

P6(Preservation of ordering on conditional events): $\pi(A \cap B)/\pi(B) \geq \pi(C \cap D)/\pi(D) \Leftrightarrow A|B \succsim C|D$ for all A|B, C|D ∈ $\mathcal{D}$.[7]

Let us consider an example of the sample spaces and likelihood relation:

Example 1: Set S_1 = { a, b }, S_2 = { c, d }, and S = { (a, c), (a, d), (b, c), (b, d) }. Let π^1 be a real-valued function defined on $\mathfrak{B}_1 \cup \mathfrak{B}_2 \cup \mathfrak{B}_S \cup \mathfrak{B}_T$ such that: $\pi^1(a) = \pi^1(b) = \pi^1(c) = \pi^1(d) = 1/2$; $\pi^1(a, c) = \pi^1(a, d) = \pi^1(b, c) = \pi^1(b, d) = 1/4$, and that the restriction of π^1 on $\mathfrak{B}_T$ coincides with

[7] Luce (1968, Theorem 1) derives the subjective probability measure having property P6, requiring that the range of the derived probability measure is {0, 1}, {0, 1/2, 1}, or includes all rationales in [0, 1]. This section elaborates on Luce (1968, Theorem 1), dispensing with the condition for the range.

the Lebesgue measure on T. Denote $\mathcal{D}^1 \equiv \{ A \mid B \in \Gamma_1 \cup \Gamma_2 \cup \Gamma_S \cup \Gamma_T : \pi^1(B) > 0\}$, and define a real-valued function g^1 on $\mathcal{D}^1$ by

$$g^1(A \mid B) = 1/8 \qquad \text{if } A \mid B \in \Gamma_S \text{ and } B \neq S$$

$$= \pi^1(A \cap B)/\pi^1(B) \qquad \text{otherwise.}$$

Define a binary relation $\succsim^1$ on $\mathcal{D}^1$ by

$$A \mid B \succsim^1 C \mid D \Leftrightarrow g^1(A \mid B) \geq g^1(C \mid D) \text{ for all } A \mid B,\ C \mid D \in \mathcal{D}^1.$$

Because $g^1(A \mid \eta(A)) = \pi^1(A)$ for all $A \in \mathfrak{B}_1 \cup \mathfrak{B}_2 \cup \mathfrak{B}_S \cup \mathfrak{B}_T$, it holds by Theorem 1(B) that $(\mathcal{D}^1, \succsim^1)$ satisfies the axioms A1—A5 and π^1 has the properties P1—P5, which implies that π^1 is a probability function of $(\mathcal{D}^1, \succsim^1)$. Setting $A^* = \{ (a, c), (a, d), (b, c) \}$, it holds that

$$\{ (a, c) \} \mid A^* \sim^1 \{(a, c), (a, d)\} \mid A^* \text{ and}$$

$$\pi^1(\{(a, c), (a, d)\} \cap A^*)/\pi^1(A^*) = [\pi^1(a, c) + \pi^1(a, d)]/\pi^1(A^*) = 2/3$$

$$> \pi^1(\{(a, c)\} \cap A^*)/\pi^1(A^*) = \pi^1(a, c)/\pi^1(A^*) = 1/3,$$

which means that π^1 does not have the property P6.

To exclude such a likelihood relation, we introduce some additional axioms for the likelihood relation $(\mathcal{D}, \succsim)$:

A6(Consistency): For all $A, B, C, D \in \mathfrak{B}_1 \cup \mathfrak{B}_2 \cup \mathfrak{B}_S \cup \mathfrak{B}_T$ such that $A \subset B$, $C \subset D$ and $B \sim D \succ \emptyset_T$, $A \succ C \Rightarrow A \mid B \succ C \mid D$ holds, and $A \sim C \Rightarrow A \mid B \sim C \mid D$ holds.

A7(Homogeneity in Γ_T): For all $A, B \in \mathfrak{B}_T^*$ with $A \subset B$ and all $c \in (0, 1)$, if $A \mid B \in \Gamma_T \cap \mathcal{D}$ and $c \cdot A \mid c \cdot B \in \Gamma_T \cap \mathcal{D}$, then $A \mid B \sim c \cdot A \mid c \cdot B$.[8]

A8(Essentiality): $A \mid B \sim (A \cap B) \mid B$ for all $A \mid B \in \mathcal{D}$.[9]

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[8] A closely related axiom to A7 is introduced by Miyake (2016, Homogeneity) to characterize the logarithmically homogeneous preferences, which describe the difference comparisons in $\mathbb{R}^n$.

[9] The axiom A8 is introduced by Luce (1968).

The three axioms A6, A7, and A8 are standard. We require a definition to state the theorem: For a given real-valued function π^* on $\mathfrak{B}_1 \cup \mathfrak{B}_2 \cup \mathfrak{B}_S \cup \mathfrak{B}_T$, a pair $(\mathcal{D}^*, \succsim^*)$ is called the *induced relation of π^** if and only if

$$\mathcal{D}^* = \{ A \mid B \in \Gamma_1 \cup \Gamma_2 \cup \Gamma_S \cup \Gamma_T : \pi^*(B) > 0 \} \text{ and}$$

$$A \mid B \succsim^* C \mid D \Leftrightarrow \pi^*(A \cap B)/\pi^*(B) \geq \pi^*(C \cap D)/\pi^*(D) \text{ for all } A \mid B,\ C \mid D \in \mathcal{D}^*. \quad (1)$$

Then we have the following theorem:

Theorem 2:(**A**) If a likelihood relation $(\mathcal{D}, \succsim)$ satisfies A1−A8, then a probability function of $(\mathcal{D}, \succsim)$ having property P6 exists:

$$\pi(A \cap B)/\pi(B) \geq \pi(C \cap D)/\pi(D) \Leftrightarrow A \mid B \succsim C \mid D \text{ for all } A \mid B, C \mid D \in \mathcal{D}.$$

(**B**) Suppose that a real-valued function π^* exists on $\mathfrak{B}_1 \cup \mathfrak{B}_2 \cup \mathfrak{B}_S \cup \mathfrak{B}_T$ having the properties P1−P4, then the induced relation of π^* determined by (1) is a likelihood relation satisfying A1−A8. (**C**) Each of the three axioms A6, A7, and A8 is independent of the other axioms in Theorem 2(A).

Theorem 2 is proved in Section 6 of this paper. Theorem 2(A, B) implies that the eight axioms A1−A8 are necessary and sufficient for the likelihood relation on the conditional events to be represented by the conditional probability measure. Theorem 2(C) implies that each of the three axioms A6, A7, and A8 is indispensable for Theorem 2(A).

5. A theorem for the multiplicative law of subjective probability

For a given likelihood relation $(\mathcal{D}, \succsim)$ satisfying the axioms A1−A8, it holds by Theorem 2(A) that there exists a probability function of $(\mathcal{D}, \succsim)$ having the property P6. This section considers when the probability function also has the following property:

P7(Multiplicative law): $\pi(A) \cdot \pi(B) = \pi(A \times B)$ for all $A \in \mathfrak{B}_1$ and all $B \in \mathfrak{B}_2$.[10]

[10] This assertion means that $\pi(A \mid S_1) \cdot \pi(B \mid S_2) = \pi(A \times B \mid S_1 \times S_2)$ for all $A \in \mathfrak{B}_1$ and all $B \in \mathfrak{B}_2$.

Let us consider an example of the sample spaces and likelihood relation:

Example 2 (Itô, 1984, Section 1.3c): Set $S_1 = \{ H_1, T_1 \}$, $S_2 = \{ H_2, T_2 \}$, and $S = \{ (H_1, H_2), (H_1, T_2), (T_1, H_2), (T_1, T_2) \}$, and let π^2 be a real-valued function defined on $\mathfrak{B}_1 \cup \mathfrak{B}_2 \cup \mathfrak{B}_S \cup \mathfrak{B}_T$ such that $\pi^2(H_1) = \pi^2(T_1) = \pi^2(H_2) = \pi^2(T_2) = 1/2$ and $\pi^2(H_1, H_2) = \pi^2(T_1, T_2) = 3/8$, $\pi^2(H_1, T_2) = \pi^2(T_1, H_2) = 1/8$, where the restriction of π^2 on $\mathfrak{B}_T$ coincides with the Lebesgue measure on T. Let $(\mathcal{D}^2, \succsim^2)$ be the induced relation of π^2. Then, π^2 being a probability function of $(\mathcal{D}^2, \succsim^2)$ holds, and that π^2 has property P6. However, $\pi^2(H_1)\cdot\pi^2(H_2) = 1/4 < \pi^2(H_1, H_2) = 3/8$ holds, implying that π^2 does not have property P7.

To exclude such a likelihood relation, we introduce some additional axioms for the likelihood relation $(\mathcal{D}, \succsim)$:

A9(Independence of irrelevant experiment): $A \mid S_1 \sim A \times S_2 \mid S_1 \times S_2$ holds for all $A \in \mathfrak{B}_1$, and $B \mid S_2 \sim S_1 \times B \mid S_1 \times S_2$ holds for all $B \in \mathfrak{B}_2$.

A10(Independence of irrelevant events): Let A be an event in $\mathfrak{B}_2$. $S_1 \times A \mid S_1 \times S_2 \sim S_1 \times A \mid B \times S_2$ holds for all $B \in \mathfrak{B}_1$ with $B \succ \emptyset_T$.[11]

Axiom A9 requires that the likelihood of an event $A \mid S_1$ in Experiment 1 coincides with that of the (produced) event $A \times S_2 \mid S_1 \times S_2$ in Experiment 3, because S_2 is redundant when one specifies the event $A \times S_2$ conditioned on $S_1 \times S_2$.

Axiom A10 requires that the likelihood of the (producted) event $S_1 \times A$ in Experiment 3 is independent of any conditioning events in Experiment 1. Namely, A10 is the re-statement of Luce and Narens' (1978, Theorem 3) condition that $A \sim A \mid B$ in our setting. In the coin-tossing experiment as in Example 2, where a coin is tossed twice, A10 requires that

[11] $A \times S_2 \mid S_1 \times S_2 \sim A \times S_2 \mid S_1 \times B$ holds for all $A \in \mathfrak{B}_1$ and all $B \in \mathfrak{B}_2$ with $B \succ \emptyset_T$ under A1−A10, because Theorem 3 holds under A1−A10 and we have that $\pi(A) = \pi(A \times B)/\pi(B)$, implying $A \times S_2 \mid S_1 \times S_2 \sim A \times S_2 \mid S_1 \times B$.

the decision-maker's likelihood of an event in the second tossing conditioned on the outcome of the first tossing is invariant whichever outcome is drived at the first tossing. Specifically, the following holds:

$$\pi^2(H_1,H_2)/[\pi^2(H_1,H_2)+\pi^2(H_1,T_2)] = 3/4 > \pi^2(T_1,H_2)/[\pi^2(T_1,H_2)+\pi^2(T_1,T_2)] = 1/4, \qquad (2)$$

which implies that $S_1\times\{H_2\}\,|\,\{H_1\}\times S_2 \succ^2 S_1\times\{H_2\}\,|\,\{T_1\}\times S_2$.[12] Because axiom A10 requires that $S_1\times\{H_2\}\,|\,\{H_1\}\times S_2 \sim^2 S_1\times\{H_2\}\,|\,\{T_1\}\times S_2$, the likelihood relation $(\mathcal{D}^2, \succsim^2)$ does not satisfy A10, and A10 excludes $(\mathcal{D}^2, \succsim^2)$.

As a main result of this paper, we have the following theorem:

Theorem 3: (**A**) If a likelihood relation $(\mathcal{D}, \succsim)$ satisfies all through the axioms A1−A10, then there exists a probability function of $(\mathcal{D}, \succsim)$ having the properties P1−P7. (**B**) Suppose that a real-valued function π^* exists on $\mathfrak{B}_1\cup\mathfrak{B}_2\cup\mathfrak{B}_S\cup\mathfrak{B}_T$ having properties P1−P4 and P7, then the induced relation of π^* determined by (1) is a likelihood relation satisfying all through axioms A1−A10. (**C**) Each of the two axioms A9 and A10 is independent of the other axioms in Theorem 3(A).

Theorem 3 is proved in the next section. Theorem 3(A, B) implies that the ten axioms A1−A10 are necessary and sufficient for the likelihood relation to be represented by the probability satisfying the multiplicative law. Theorem 3(C) implies that each of the two axioms, A9 and A10 is indispensable for Theorem 3(A).

6. Proof of the theorems

Proof of Theorem 1: (**A**) Suppose that a likelihood relation $(\mathcal{D}, \succsim)$ satisfies A1−A5. We will prove that there exists a real-valued function π on $\mathfrak{B}_1\cup\mathfrak{B}_2\cup\mathfrak{B}_S\cup\mathfrak{B}_T$ having the properties

[12] This can be recognized as a result of the decision-maker's updating of the likelihood based on the outcome of the first coin-tossing, if the decision-maker follows the maximal likelihood principle.

P1−P5. We need a lemma, which is proved in Appendix:

Lemma 2: If($\mathcal{D}$, $\succsim$)satisfies A1−A5, then the following ten assertions hold: (**i**) $\{ a \} \equiv [a, a] \sim \emptyset_T$ for all $a \in T$. (**ii**) $[a, b] \succ \emptyset_T$ for all $a, b \in T$ with $a < b$. (**iii**) $[a, b] \sim [a, b) \sim (a, b] \sim (a, b)$ for all $a, b \in T$ with $a < b$. (**iv**) $[0, a] \succsim [0, b] \Leftrightarrow a \geq b$ for all $a, b \in T$. (**v**) $\mu(J) \geq \mu(K) \Leftrightarrow J \succsim K$ for all $J, K \in \mathfrak{B}_T^*$. (**vi**) $A \succsim B \Leftrightarrow B^C \succsim A^C$ for all $A, B \in \mathfrak{B}_1 \cup \mathfrak{B}_2 \cup \mathfrak{B}_S \cup \mathfrak{B}_T$ with $\eta(A) = \eta(B)$, where $A^C \equiv \{ s \in \eta(A): s \notin A \}$. (**vii**) Let $\{ B_n \}$ be a sequence of events in $\mathfrak{B}_1 \cup \mathfrak{B}_2 \cup \mathfrak{B}_S \cup \mathfrak{B}_T$. If $B_n \subset B_{n+1}$ for all n, and if there exists $A \in \mathfrak{B}_1 \cup \mathfrak{B}_2 \cup \mathfrak{B}_S \cup \mathfrak{B}_T$ such that $A \succsim B_n$ for all n, then $A \succsim \cup_n B_n$, where $\cup_n B_n \equiv \bigcup_{n=1}^{\infty} B_n$. (**viii**) For any $A \in \mathfrak{B}_1 \cup \mathfrak{B}_2 \cup \mathfrak{B}_S \cup \mathfrak{B}_T$, $\{ x \in T : [0, x] \succsim A \}$ is non-empty and closed in T. (**ix**) For any $A \in \mathfrak{B}_1 \cup \mathfrak{B}_2 \cup \mathfrak{B}_S \cup \mathfrak{B}_T$, $\{ x \in T : A \succsim [0, x] \}$ is non-empty and closed in T. (**x**) For any $A \in \mathfrak{B}_1 \cup \mathfrak{B}_2 \cup \mathfrak{B}_S \cup \mathfrak{B}_T$ there uniquely exists a real number $x \in T$ such that $A \sim [0, x]$.

Define a real-valued function π on $\mathfrak{B}_1 \cup \mathfrak{B}_2 \cup \mathfrak{B}_S \cup \mathfrak{B}_T$ by

$$\pi(A) = x \text{ for all } A \in \mathfrak{B}_1 \cup \mathfrak{B}_2 \cup \mathfrak{B}_S \cup \mathfrak{B}_T, \tag{3}$$

where x is the number determined uniquely by Lemma 2(x) for each $A \in \mathfrak{B}_1 \cup \mathfrak{B}_2 \cup \mathfrak{B}_S \cup \mathfrak{B}_T$.

(**π has property P5**): For all $A, B \in \mathfrak{B}_1 \cup \mathfrak{B}_2 \cup \mathfrak{B}_S \cup \mathfrak{B}_T$, it holds by (3) and Lemma 2(v) that $A \succsim B \Leftrightarrow [0, \pi(A)] \succsim [0, \pi(B)] \Leftrightarrow \pi(A) \geq \pi(B)$, which means that π has property P5.

(**π has property P3**): We have by (3) and A1 that $\pi(S) = 1$, and we have by (3), A1 and Lemma 2(i) that $\pi(\emptyset_S) = 0$. We prove that π is finitely additive on $\mathfrak{B}_S$. Fix any $A, B \in \mathfrak{B}_S$ with $A \cap B = \emptyset_S$. It holds by (3) that $A \sim [0, \pi(A)]$ and $B \sim [0, \pi(B)]$. It follows from Lemma 2(iii, v) that $A \sim [0, \pi(A)] \sim [0, \pi(A))$ and $B \sim [0, \pi(B)] \sim [\pi(A), \pi(A)+\pi(B)]$. We have by A2 that $A \cup B \sim [0, \pi(A)) \cup [\pi(A), \pi(A)+\pi(B)] = [0, \pi(A)+\pi(B)]$, which implies that $\pi(A \cup B) = \pi(A)+\pi(B)$. Hence π is finitely additive on $\mathfrak{B}_S$.

We prove that π is σ-additive on $\mathfrak{B}_S$. We need a lemma proved in Appendix:

Lemma 3: If $\{ A_n \}$ is a sequence of events in $\mathfrak{B}_S$ satisfying $A_{n+1} \subset A_n$ for all n, and if $\cap_n A_n$

$= \emptyset_S$, then $\lim \pi(A_n) = 0$.

It holds by Itô (1984, Ch.2 Theorem 2.2.4 and Remark) and Lemma 3 that π is σ-additive on $\mathfrak{B}_S$. Thus π has property P3.

(**π has properties P1 and P2**): We can prove that π has properties P1 and P2, using almost the same manner as the proof in case of P3.

(**π has property P4**): It holds by (3) and Lemma 2(i) that $\pi(T) = 1$ and $\pi(\emptyset_T) = 0$. Fix any $A, B \in \mathfrak{B}_T$ with $A \cap B = \emptyset$. We can prove that π is finitely additive on $\mathfrak{B}_T$ by almost the same manner in the proof above. For any intervals $J \in \mathfrak{B}_T^*$, we have $J \sim [0, \mu(J)]$ by Lemma 2(v), which implies $\pi(J) = \mu(J)$. Hence, we have by Hopf's extension theorem (Itô, 1984, Ch.2 Theorem 2.2.4) that $\pi(A) = \mu(A)$ for all $A \in \mathfrak{B}_T$, because μ is σ-additive on $\mathfrak{B}_T$.

(**B**) Suppose that there exists a real-valued function π^* on $\mathfrak{B}_1 \cup \mathfrak{B}_2 \cup \mathfrak{B}_S \cup \mathfrak{B}_T$ having properties P1−P4. Let $(\mathcal{D}^*, \succsim^*)$ be the induced relation of π^*. Then we can prove easily that $(\mathcal{D}^*, \succsim^*)$ satisfies all the axioms. □

Proof of Theorem 2: (**A**) Suppose that a likelihood relation $(\mathcal{D}, \succsim)$ satisfies all the axioms. It folllows from Theorem 1 that there exists a real-valued function π having properties P1−P5. We will prove that π has property P6. We need two lemmas proved in Appendix:

Lemma 4: (**i**) Fix any $A_2 | A_1, A_4 | A_3, B_2 | B_1, B_4 | B_3 \in \mathcal{D}$. If $A_i \sim B_i$ for $i = 1, 2, 3, 4$, and if $A_{i+1} \subset A_i$ and $B_{i+1} \subset B_i$ for i = 1, 3, then $A_2 | A_1 \succsim A_4 | A_3 \Leftrightarrow B_2 | B_1 \succsim B_4 | B_3$. (**ii**) For all A, B, C, $D \in \mathfrak{B}_1 \cup \mathfrak{B}_2 \cup \mathfrak{B}_S \cup \mathfrak{B}_T$ such that $A \subset B$, $C \subset D$ and $B \sim D \succ \emptyset_T$, $C | D \succsim A | B \Rightarrow C \succsim A$.

Lemma 5: (**i**) $[0, \beta] | [0, \alpha] \succsim [0, \delta] | [0, \gamma] \Leftrightarrow \beta/\alpha \geq \delta/\gamma$ for all $\alpha, \beta, \gamma, \delta \in T$ such that $\alpha, \gamma > 0$, $\alpha \geq \beta$, $\gamma \geq \delta$. (**ii**) $A | B \succsim C | D \Leftrightarrow \pi(A)/\pi(B) \geq \pi(C)/\pi(D)$ for any $A | B, C | D \in \mathcal{D}$ with $A \subset B$ and $C \subset D$.

Fix any $A | B, C | D \in \mathcal{D}$. It holds by A8 and Lemma 5(ii) that

$$A | B \succsim C | D \Leftrightarrow (A \cap B) | B \succsim (C \cap D) | D \Leftrightarrow \pi(A \cap B)/\pi(B) \geq \pi(C \cap D)/\pi(D),$$

which menas that π has property P6.

(**B**) Suppose that a real-valued function π^* exists on $\mathfrak{B}_1 \cup \mathfrak{B}_2 \cup \mathfrak{B}_S \cup \mathfrak{B}_T$ having properties P1−P4. Let $(\mathcal{D}^*, \succsim^*)$ be the induced relation of π^*. Then we can prove easily that $(\mathcal{D}^*, \succsim^*)$ satisfies A1−A8.

(**C**) **(Independency of A6)**: Let us consider Example 1 again. It suffices to prove that $(\mathcal{D}^1, \succsim^1)$ does not satisfy A6 and $(\mathcal{D}^1, \succsim^1)$ satisfies A7 and A8. Let π^1 be the symmetric probability distribution defined in Example 1. Because

$$\pi^1(\{(a, c), (a, d)\}) = \pi^1(a, c) + \pi^1(a, d) = 1/2 > \pi^1(a, c) = 1/4,$$

implies $\{(a, c), (a, d)\} \succ^1 \{(a, c)\}$, it holds that

$$\{(a, c), (a, d)\} \succ^1 \{(a, c)\} \text{ and } \{(a, c)\} \mid A^* \sim^1 \{(a, c), (a, d)\} \mid A^*,$$

which implies that $(\mathcal{D}^1, \succsim^1)$ does not satisfy A6. Because the restriction of π^1 on $\mathfrak{B}_T$ coincides with the Lebesgue measure on T, $(\mathcal{D}^1, \succsim^1)$ satisfies A7. Set $\Gamma^* = \{ A \mid B \in \Gamma_S : B \neq S \}$. For $A \mid B \in \mathcal{D}^1$, let us consier the two cases:

Case 1($A \mid B \in \Gamma^*$): Because $(A \cap B) \mid B \in \Gamma^*$, we have by the definition of $\succsim^1$ that $A \mid B \sim^1 (A \cap B) \mid B$.

Case 2($A \mid B \notin \Gamma^*$): Because $(A \cap B) \mid B \in \Gamma^*$, we have that $g^1(A \mid B) = \pi^1(A \cap B)/\pi^1(B)$ and $g^1(A \cap B) \mid B) = \pi^1(A \cap B)/\pi^1(B)$, which implies $A \mid B \sim^1 (A \cap B) \mid B$.

Hence we have that $(\mathcal{D}^1, \succsim^1)$ satisfies A8.

(Independency of A7): Let us consider $S_1 = \{a, b\}$, $S_2 = \{c, d\}$, and $S = \{ (a, c), (a, d), (b, c), (b, d) \}$ again. Let π^1 be the symmetric probability distribution defined in Example 1. Define $\mathcal{D}^3$ by $\mathcal{D}^3 = \mathcal{D}^1$, and define a real-valued function g^3 on $\mathcal{D}^3$ by

$$
\begin{aligned}
g^2(A \mid B) &= (4/5)\cdot[\pi^1(A \cap B)/\pi^1(B)] + (1/5) && \text{if } A \mid B \in \Gamma_T \text{ and } \pi^1(A \cap B)/\pi^1(B) > 7/12 \\
&= (1/12)\cdot\pi^1(B) + (13/24) && \text{if } A \mid B \in \Gamma_T \text{ and } \pi^1(A \cap B)/\pi^1(B) = 7/12 \\
&= (6/7)\cdot[\pi^1(A \cap B)/\pi^1(B)] && \text{if } A \mid B \in \Gamma_T \text{ and } \pi^1(A \cap B)/\pi^1(B) < 7/12
\end{aligned}
$$

$= \pi^1(A\cap B)/\pi^1(B) \qquad\qquad \text{if } A\,|\,B \in \Gamma_1\cup\Gamma_2\cup\Gamma_S$.

Define a binary relation $\succsim^3$ on $\mathcal{D}^3$ by $A\,|\,B \succsim^3 C\,|\,D \Leftrightarrow g^3(A\,|\,B) \geq g^3(C\,|\,D)$ for all $A\,|\,B,\ C\,|\,D \in \mathcal{D}^3$. Then it holds that $[0, 7/24]\,|\,[0, 1/2] = 2\cdot[0, 7/48]\,|\,2\cdot[0, 1/4] \succ^3 [0, 7/48]\,|\,[0, 1/4]$, which means that $(\mathcal{D}^3, \succsim^3)$ does not satisfy A7. Moreover, it holds that

$$\pi^1([0, 7/24]\cap[0, 1/2])/\pi^1([0, 1/2]) = 7/12 = \pi^1([0, 7/48]\cap[0, 1/4])/\pi^1([0, 1/4]),$$

which means that π^1 does not have the property P6 with respect to $(\mathcal{D}^3, \succsim^3)$, even though $(\mathcal{D}^3, \succsim^3)$ satisfies A1−A6 and A8. Because it holds by the definition of $g^3(A\,|\,B)$ that $g^3(A\,|\,B) = g^3((A\cap B)\,|\,B)$ for all $A\,|\,B \in \mathcal{D}^3$, we have that $A\,|\,B \sim^3 (A\cap B)\,|\,B$, which means that $(\mathcal{D}^3, \succsim^3)$ satisfies A8. For all $A, B, C, D \in \mathfrak{B}_1\cup\mathfrak{B}_2\cup\mathfrak{B}_S\cup\mathfrak{B}_T$ such that $A \subset B$, $C \subset D$ and $B \sim D \succ \emptyset_T$, it holds that $A \succ C \Rightarrow g^3(A\,|\,B) > g^3(C\,|\,D)$, and that $A \sim C \Rightarrow g^3(A\,|\,B) = g^3(C\,|\,D)$. Hence $(\mathcal{D}^3, \succsim^3)$ satisfies A6.

(Independency of A8): Let us consider $S_1 = \{a, b\}$, $S_2 = \{c, d\}$, and $S = \{ (a, c), (a, d), (b, c), (b, d) \}$ again. Let π^1 be the symmetric probability distribution defined in Example 1. Define $\mathcal{D}^4$ by $\mathcal{D}^4 = \mathcal{D}^1$, and efine a binary relation $\succsim^4$ on $\mathcal{D}^4$ by

$$A\,|\,B \succsim^4 C\,|\,D \;\Leftrightarrow\; \pi^1(A)/\pi^1(B) \geq \pi^1(C)/\pi^1(D) \text{ for all } A\,|\,B,\ C\,|\,D \in \mathcal{D}^4.$$

Then it holds that $[0, 5/8]\,|\,[1/2, 1] \succ^4 [1/2, 5/8]\,|\,[1/2, 1]$, which means that $(\mathcal{D}^4, \succsim^4)$ does not satisfy A8. Moreover, it holds that

$$\pi^1([0, 5/8]\cap[1/2, 1])/\pi^1([1/2, 1]) = 1/4 = \pi^1([1/2, 5/8]\cap[1/2, 1])/\pi^1([1/2, 1]).$$

Hence π^1 does not have property P6 with respect to $(\mathcal{D}^4, \succsim^4)$, even though $(\mathcal{D}^4, \succsim^4)$ satisfies A1−A7. □

Proof of Theorem 3: (**A**) Suppose that a likelihood relation $(\mathcal{D}, \succsim)$ satisfies all the axioms. It folllows from Theorem 2 that there exists a real-valued function π having properties P1−P6. We will prove tha π has property P7. Fix any $A\in \mathfrak{B}_1$ and $A\in \mathfrak{B}_2$.

Case 1($A \succ \emptyset_T$): It holds by A10 that

$$\pi(A) = \pi(A\times S_2) \text{ and } \pi(B) = \pi(S_1\times B) \quad (4)$$

It holds by A9 and Theorem 2 that $\pi(S_1\times B) = \pi(S_1\times B)/\pi(S_1\times S_2) = \pi(A\times B)/\pi(A\times S_2)$. Hence we have by (4) and this that $\pi(B) = \pi(S_1\times B) = \pi(A\times B)/\pi(A\times S_2) = \pi(A\times B)/\pi(A)$, which implies that $\pi(A\times B) = \pi(A)\pi(B)$.

Case 2($A \sim \emptyset_T$): It holds by Theorem 1 that $\pi(A) = 0$. It holds by A10 that $\pi(A\times S_2)$. Because $A\times B \subset A\times S_2$, we have that $\pi(A\times B) = 0$. Hence $\pi(A)\pi(B) = \pi(A\times B) = 0$.

(**B**) Suppose that a real-valued function π^* exists on $\mathfrak{B}_1\cup\mathfrak{B}_2\cup\mathfrak{B}_S\cup\mathfrak{B}_T$ having the properties P1−P4 and P7. Let $(\mathcal{D}^*, \succsim^*)$ be the induced relation of π^*. Then we can prove easily that $(\mathcal{D}^*, \succsim^*)$ satisfies A1−A10.

(**C**) **(Independency of A9)**: Let us consider $S_1 = \{a, b\}$, $S_2 = \{c, d\}$, and $S = \{ (a, c), (a, d), (b, c), (b, d) \}$ again. Let π^3 be a real-valued function defined on $\mathfrak{B}_1\cup\mathfrak{B}_2\cup\mathfrak{B}_S\cup\mathfrak{B}_T$ such that: $\pi^3(a) = 1/4$, $\pi^3(b) = 3/4$; $\pi^3(c) = \pi^3(d) = 1/2$; $\pi^3(a, c) = \pi^3(a, d) = \pi^3(b, c) = \pi^3(b, d) = 1/4$, and that the restriction of π^3 on $\mathfrak{B}_T$ coincides with the Lebesgue measure μ on T. Hence it holds that

$$\pi^3(a)\cdot\pi^3(S_2) = 1/4 < \pi^3(\{ a \}\times S_2) = \pi^3(\{ a \}\times\{c, d\})$$

$$= \pi^3(\{(a, c)\}\cup\{(a, d)\}) = \pi^3(a, c) + \pi^3(a, d) = 1/2,$$

which means that π^3 does not have the property P7. Moreover, let $(\mathcal{D}^5, \succsim^5)$ be the induced relation of π^3. Namely, $(\mathcal{D}^5, \succsim^5)$ is determined by (1). Then it holds that

$$\pi^3(a) = \pi^3(a \mid S_1) = 1/4 \text{ and } \pi^3(\{ a \}\times S_2) = \pi^3(\{ a \}\times S_2 \mid S_1\times S_2) = 1/2,$$

which implies that $\{ a \}\times S_2 \mid S_1\times S_2 \succ^5 a \mid S_1$. Hence $(\mathcal{D}^5, \succsim^5)$ does not satisfy A9. Because π^3 has properties P1−P6, it holds by Theorem 2(B) that $(\mathcal{D}^5, \succsim^5)$ satisfies A1−A8. Moreover, it holds that

$$\pi^3(\{ a \}\times S_2) = \pi^3(a, c) + \pi^3(a, d) = 1/2; \pi^3([\{ a \}\times\{ c \}]\cap[S_1\times\{ c \}])/\pi^3(S_1\times\{ c \})$$

$$= \pi^3(a, c)/[\pi^3(a, c)+\pi^3(b, c)] = (1/4)/(1/2) = 1/2,$$

$$\pi^3(\{ a \}\times S_2) = \pi^3(a, c) + \pi^3(b, d) = 1/2; \pi^3([\{ a \}\times\{ d \}]\cap[S_1\times\{ d \}])/\pi^3(S_1\times\{ d \})$$

$= \pi^3(a, d)/[\pi^3(a, d)+ \pi^3(b, d)] = (1/4)/(1/2) = 1/2,$

$\pi^3(\{ b \}\times S_2) = \pi^3(b, c) + \pi^3(b, d) = 1/2; \pi^3([\{ b \}\times\{ c \}]\cap[S_1\times\{ c \}])/\pi^3(S_1\times\{ c \})$

$= \pi^3(b, c)/[\pi^3(a, c) + \pi^3(b, c)] = (1/4)/(1/2) = 1/2,$

$\pi^3(\{ b \}\times S_2) = \pi^3(b, c) + \pi^3(b, d) = 1/2; \pi^3([\{ b \}\times\{ d \}]\cap[S_1\times\{ d \}])/\pi^3(S_1\times\{ d \})$

$= \pi^3(b, d)/[\pi^3(a, d) + \pi^3(b, d)] = (1/4)/(1/2) = 1/2,$

which implies that $(\mathcal{D}^5, \succsim^5)$ satisfies the axioms A10.

(Independency of A10): Let us consider Example 2 again. Because it follows from (2) that $(\mathcal{D}^2, \succsim^2)$ does not satisfy A10, it suffices to prove that $(\mathcal{D}^2, \succsim^2)$ satisfies A1−A9. Let π^2 be the probability distribution defined in Example 2. Because π^2 has properties P1−P6, it holds by Theorem 2(B) that $(\mathcal{D}^2, \succsim^2)$ satisfies A−A8. Moreover, it holds that

$\pi^2(\{ H_1 \}\times S_2) = \pi^2(H_1, H_2) + \pi^2(H_1, T_2) = 1/2 = \pi^2(H_1);$

$\pi^2(\{ T_1 \}\times S_2) = \pi^2(T_1, H_2) + \pi^2(T_1, T_2) = 1/2 = \pi^2(T_1);$

$\pi^2(S_1\times\{ H_2 \}) = p^2(H_1, H_2) + p^2(T_1, H_2) = 1/2 = p^2(H_2);$

$\pi^2(S_1\times\{ T_2 \}) = \pi^2(H_1, T_2) + \pi^2(T_1, T_2) = 1/2 = \pi^2(T_2),$

which implies that $(\mathcal{D}^2, \succsim^2)$ satisfies A9. □

Appendix

Proof of Lemma 2: (**i**) It holds by A1 and A4 that $\{ a \} \sim \emptyset_T$ for all $a \in [0, 1]$. (**ii**) Fix any $a, b \in [0, 1]$ with $b > a$, it holds by A4 that $[a, b] \succ \emptyset_T$. (**iii**) Fix any $a, b \in [0, 1]$ with $a < b$. Because $[a, b) \succsim [a, b)$ and $\{ b \} \succsim \emptyset_T$ by A1, it holds by A2 that $[a, b] \succsim [a, b)$. Because $[a, b) \succsim [a, b)$ and $\emptyset_T \succsim \{ b \}$ by Lemma 2(i), it holds by A2 that $[a, b) \succsim [a, b]$. Hence $[a, b] \sim [a, b)$. By almost the same manner we can prove that $[a, b] \sim (a, b] \sim (a, b)$. (**iv**) Suppose that $a \geq b$. Because $[0, b) \succsim [0, b)$ and $[b, a] \succsim \emptyset_T$ by A1, it holds by A2 that $[0, a] \succsim [0, b)$. It holds by Lemma 2(iii) that $[0, a] \succsim [0, b) \sim [0, b]$. Suppose that $b > a$. Because $[0, a) \succsim [0, a)$ and $[a, b] \succ \emptyset_T$ by Lemma 2(ii), it holds by A2 that $[0, b] \succ [0, a)$. It holds by Lemma

2(iii) that $[0, b] \succ [0, a) \sim [0, a]$. Hence we have that $b > a \Rightarrow [0, b] \succ [0, a]$, which implies $[0, a] \succsim [0, b] \Rightarrow a \geq b$. (**v**) It follows from Lemma 2(iii) that it suffices to prove the case of the closed intervals. For any intervals $[a, b], [c, d] \in \mathfrak{B}_{T^*}$, it holds by A5 that $[0, b - a] \sim [a, b]$ and $[0, d - c] \sim [c, d]$. Hence we have by Lemma 2(iv) that $\mu([a, b]) \geq \mu([c, d]) \Leftrightarrow (b - a) \geq (d - c) \Leftrightarrow [0, b - a] \succsim [0, d - c] \Leftrightarrow [a, b] \succsim [c, d]$. (**vi**) Assume that $A \succsim B$. If $A^C \succ B^C$, it holds by $A \succsim B$ and A2 that $[A^C \cup A] \succ [B^C \cup B]$, which is a contradiction. Thus we have that $B^C \succsim A^C$. (**vii**) Suppose that $B_n \subset B_{n+1}$ for all n and that there exists $A \in \mathfrak{B}_1 \cup \mathfrak{B}_2 \cup \mathfrak{B}_S \cup \mathfrak{B}_T$ such that $A \succsim B_n$ for all n. Define $C_n = B_n^C$ for all n, and define $D = A^C$. Then it holds by Lemma 2(vi) that $C_{n+1} \subset C_n$ for all n and that $C_n \succsim D$ for all n. Hence we have by A3 that $\cap_n C_n \succsim D$, which implies that $A = D^C \succsim (\cap_n C_n)^C = \cup_n B_n$. (**viii**) Fix any $A \in \mathfrak{B}_1 \cup \mathfrak{B}_2 \cup \mathfrak{B}_S \cup \mathfrak{B}_T$. It holds by A1 that $T \succsim A$, which implies that $1 \in \{ x \in T : [0, x] \succsim A \} \neq \emptyset_T$. Suppose that a sequence $\{ x_n \}$ in T converges to x^* and that $[0, x_n] \succsim A$ for all n. It suffices to prove that $x^* \in T$ and $[0, x^*] \succsim A$. Because T is closed, it holds that $x^* \in T \neq \emptyset_T$. Define $\{ B_n \}$ in $\mathfrak{B}_{T^*}$ by $B_n = [0, x_n]$ for all n. It holds by Anderson and Hall (1963, Ch II, Theorem 1.10 and Ch. IV, Theorem 1.9) that we can assume that $x_n \geq x_{n+1}$ for all n, or that $x_n \leq x_{n+1}$ for all n, without loss of generality.

Case 1($x_n \geq x_{n+1}$ for all n): We have $B_{n+1} \subset B_n$ for all n by $x_n \geq x_{n+1}$ for all n. Because $B_n \succsim A$ for all n, we have by A3 that $\cap_n B_n \succsim A$. Hence it holds by $\cap_n B_n \subset [0, x^*]$ and Lemma 2(iv) that $[0, x^*] \succsim A$.

Case 2($x_n \leq x_{n+1}$ for all n) : We have $B_n \subset B_{n+1}$ for all n by $x_n \leq x_{n+1}$ for all n. Because $B_1 \succsim A$ and $B_1 \subset \cap_n B_n \subset [0, x^*]$, it holds by Lemma 2(iv) that $[0, x^*] \succsim A$.

(**ix**) Fix any $A \in \mathfrak{B}_1 \cup \mathfrak{B}_2 \cup \mathfrak{B}_S \cup \mathfrak{B}_T$. It holds by A1 and Lemma 2(i) that $A \succsim [0, 0]$, which implies that $0 \in \{ x \in T : A \succsim [0, x] \} \neq \emptyset_T$. Suppose that a sequence $\{ x_n \}$ in T converges to x^* and that $A \succsim [0, x_n]$ for all n. It suffices to prove that $x^* \in T$ and $A \succsim [0, x^*]$. Because

T is closed, it holds that $x^* \in T \neq \emptyset_T$. Define $\{ B_n \}$ in $\mathfrak{B}_{T^*}$ by $B_n = [0, x_n]$ for all n. It holds by Anderson and Hall (1963, Ch II, Theorem 1.10 and Ch. IV, Theorem 1.9) that we can assume that $x_n \geq x_{n+1}$ for all n, or that $x_n \leq x_{n+1}$ for all n, without loss of generality.

Case 1($x_n \geq x_{n+1}$ for all n): We have $B_{n+1} \subset B_n$ for all n by $x_n \geq x_{n+1}$ for all n. Because $A \succsim B_1$ and $[0, x^*] \subset B_1$, we have by Lemma 2(iv) that $A \succsim B_1 \succsim [0, x^*]$.

Case 2($x_n \leq x_{n+1}$ for all n): We have $B_n \subset B_{n+1}$ for all n by $x_n \leq x_{n+1}$ for all n. Because $A \succsim B_n$ for all n, it holds by Lemma 2(vii) that $A \succsim \cup_n B_n$. Hence we have by $[0, x^*) \subset \cup_n B_n$ and Lemma 2(iv, iii) that $A \succsim \cup_n B_n \succsim [0, x^*) \sim [0, x^*]$. (**x**) Fix any $A \in \mathfrak{B}_S \cup \mathfrak{B}_T$. It holds by Lemma 2(viii, ix) that the two sets $\{ x \in T: [0, x] \succsim A \}$ and $\{ x \in T: A \succsim [0, x] \}$ are non-empty and closed in T. Because it holds by Smith (1983, Ch.7, Theorem 7.4) that T is connected, we have that $\{ x \in T : [0, x] \succsim A \} \cap \{ x \in T : A \succsim [0, x] \} \neq \emptyset_T$ and there exists a real number x $\in T$ such that $A \sim [0, x]$. The uniqueness of $x \in T$ is ensured by Lemma 2(iv). □

Proof of Lemma 3: Because π is finitely additive on $\mathfrak{B}_S$ and $A_n = A_{n+1} \cup (A_n / A_{n+1})$ for all n, we have that $\pi(A_n) \geq \pi(A_{n+1}) \geq 0$ for all n, which implies that $\{ \pi(A_n) \}$ is a bounded monotone sequence. Hence it holds by Smith (1983, Ch.6, Theorem 2.2) that $\lim \pi(A_n)$ exists and $\lim \pi(A_n) \geq 0$. Suppose that $\lim \pi(A_n) > 0$, and set $L = \lim \pi(A_n) > 0$. It holds by (3) that $A_n \sim [0, \pi(A_n)]$. Because $\pi(A_n) \geq L > 0$ for all n, we have by Lemma 2(v) that $A_n \sim [0, \pi(A_n)] \succsim [0, L] \succ [0, 0] \sim \emptyset_T$ for all n. It holds by A1 and A3 that $\cap_n A_n \succ \emptyset_T \sim \emptyset_S$. This contradicts with $\cap_n A_n = \emptyset_S$. Hence we have $\lim \pi(A_n) = 0$. □

Proof of Lemma 4: (**i**) It holds by A6 that $A_2 | A_1 \sim B_2 | B_1$ and $A_4 | A_3 \sim B_4 | B_3$. Hence we have $A_2 | A_1 \succsim A_4 | A_3 \Leftrightarrow B_2 | B_1 \succsim B_4 | B_3$. (**i**) Fix any $A, B, C, D \in \mathfrak{B}_1 \cup \mathfrak{B}_2 \cup \mathfrak{B}_S \cup \mathfrak{B}_T$ such that $A \subset B$, $C \subset D$ and $B \sim D \succ \emptyset_T$. Because the contraposition of $A \succ C \Rightarrow A | B \succ C | D$ is that $C | D \succsim A | B \Rightarrow C \succsim A$, we have by A6 that $C | D \succsim A | B \Rightarrow C \succsim A$. □

Proof of Lemma 5: (**i**) **Case 1** ($\alpha > \gamma$): It holds by $1 > (\gamma/\alpha) > 0$ and A7 that

$$[0, \beta] \mid [0, \alpha] \sim [0, (\gamma/\alpha)\beta] \mid [0, (\gamma/\alpha)\, \alpha] = [0, (\gamma/\alpha)\beta] \mid [0, \gamma]. \qquad (5)$$

It holds by A6 and Lemma 4(ii) that

$$[0, (\gamma/\alpha)\beta] \succsim [0, \delta] \;\Leftrightarrow\; [0, (\gamma/\alpha)\beta] \mid [0, \gamma] \succsim [0, \delta] \mid [0, \gamma]. \qquad (6)$$

Hence we have by (5), (6) and Lemma 2(v) that

$$[0, \beta] \mid [0, \alpha] \succsim [0, \delta] \mid [0, \gamma] \;\Leftrightarrow\; [0, (\gamma/\alpha)\beta] \mid [0, \gamma] \succsim [0, \delta] \mid [0, \gamma]$$

$$\Leftrightarrow\; [0, (\gamma/\alpha)\beta] \succsim [0, \delta] \;\Leftrightarrow\; (\gamma/\alpha)\beta \geq \delta \;\Leftrightarrow\; \beta/\alpha \geq \delta/\gamma.$$

Case 2 ($\alpha < \gamma$): It holds by $0 < \alpha/\gamma < 1$ and A7 that

$$[0, \delta] \mid [0, \gamma] \sim [0, (\alpha/\gamma)\delta] \mid [0, (\alpha/\gamma)\gamma] = [0, (\alpha/\gamma)\delta] \mid [0, \alpha].$$

Hence we have by A6, Lemma 4(ii) and Lemma 2(v) that

$$[0, \beta] \mid [0, \alpha] \succsim [0, \delta] \mid [0, \gamma] \;\Leftrightarrow\; [0, \beta] \mid [0, \alpha] \succsim [0, (\alpha/\gamma)\delta] \mid [0, \alpha]$$

$$\Leftrightarrow\; [0, \beta] \succsim [0, (\alpha/\gamma)\delta] \;\Leftrightarrow\; \beta \geq (\alpha/\gamma)\delta \;\Leftrightarrow\; \beta/\alpha \geq \delta/\gamma.$$

Case 3 ($\alpha = \gamma$): We have by A6, Lemma 4(ii) and Lemma 2(v) that

$$[0, \beta] \mid [0, \alpha] \succsim [0, \delta] \mid [0, \gamma] \;\Leftrightarrow\; [0, \beta] \succsim [0, \delta] \;\Leftrightarrow\; \beta \geq \delta \;\Leftrightarrow\; \beta/\alpha \geq \delta/\gamma.$$

(**ii**): Fix any $A \mid B, C \mid D \in \mathcal{D}$ with $A \subset B$ and $C \subset D$. It holds by the definition of π and Lemmas 2 and 3(i) that

$$A \mid B \succsim C \mid D \Leftrightarrow [0, \pi(A)] \mid [0, \pi(B)] \succsim [0, \pi(C)] \mid [0, \pi(D)] \Leftrightarrow \pi(A)/\pi(B) \geq \pi(C)/\pi(D). \qquad \square$$

References

Anderson, K.W., and Hall, D.W., 1963, "*Sets, sequences and mappings*", Wiley, New York.

DeGroot, M.H., 1970, "*Optimal Statistical Decisions*", McGraw-Hill, New York.

Fishburn, P.C., 1986, 'The axioms of subjective probability, *Statistical Science* **1**, 345-355.

Fishburn, P.C., 1994, Utility and subjective probability, R.J. Aumann, S. Hart (Eds.), Handbook of Game Theory Ch.39, Vol. 2, North-Holland, Amsterdam, 1397-1435.

French, S., 1982, On the axiomatisation of subjective probabilities, *Theory and Decision* **14**, 19-33.

French, S., 2022, Axiomatizing the bayesian paradigm in parallel small worlds, *Operations Research*, **70**, 1342-1358.

Grabisch, M., Monet, B., and Vergopoulos, V., 2023, Subjective expected utility through stochastic independence, *Economic Theory* **76**, 723-757.

Itô, K., (1984) "*Introduction to probability theory*", Cambridge University Press, Cambridge.

Luce, R.D., 1968, On the numerical representation of qualitative conditional probability, *Annals of Mathematical Statistics* **39**, 481-491.

Luce, R. D., and Narens, L., 1978, Qualitative independence in probability theory, *Theory and Decision* **9**, 225 - 239.

Miyake, M., 2016, Logarithmically homogeneous preferences, *Journal of Mathematical Economics* **67**, 1-9.

Smith, K. T., 1983, "*Primer of modern analysis*", Springer-Verlag, New York.