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Discussion Paper No.465

Evolution of the prior beliefs in the simple Bayesian hypothesis tests: A selection of the testing agents with the correct beliefs

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August 24, 2022

TOHOKU ECONOMICS RESEARCH GROUP

Discussion Paper

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Evolution of the prior beliefs in the simple Bayesian hypothesis tests: A selection of the testing agents with the correct beliefs

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Abstract: In general, a successful decision maker need not have been endowed with the correct prior belief on the states of nature. This paper, however, demonstrates that a simple Bayesian hypothesis test scheme has a good property: The probability of a testing agent to obtain the optimal outcome is maximized *only* if the prior belief of the agent coincides with the "correct" one, when the agent selects the Bayesian-optimal strategy with respect to his or her (own) prior belief. Consequently, in an evolutional setting, where the Bayesian test is conducted repeatedly in parallel by many testing agents with diverse prior beliefs, if the fitness value is determined by the outcome, then only the agents endowed with the correct prior beliefs survive. This result expains why an agent's prior belief can be assumed to coincide with the correct one in the Bayesian hypothesis test, as if the agent knows the true probability that was assigned by nature.

Key words: Bayesian hypothesis test, diagnostic test, prior belief, Neyman-Pearson lemma, natural selection, Malthusian competition, Laplace's principle

1. Introduction

In Bayesian hypothesis tests of a simple hypothesis,[1] the testing agent is not informed of the true probability of the events "the null hypothesis is true" or "the alternative hypothesis is true" assigned by the nature; however the agent has the personal prior beliefs (i.e., subjective probabilities) regarding the events. Although the agent can implement the Bayesian-optimal test procedure for a given prior belief, which maximizes the (posterior) expected value of the test outcome conditioned on the sample (signal), there is one problem: At what level should such a prior belief be determined ?

One possible answer is that such prior belief coincides with the "correct" belief, because the correct prior belief can be derived from past information in general by applying the statistical inference theory or Bayesian learning theory.[2] Namely, before a test, if a sufficient number of repetitions of the same test have been conducted, and if the prior belief is revised based upon the sample (or signal) at each repetition, it is possible that the sequence of revised prior beliefs will converge to the correct one.

Such a long-term learning process can be conducted when the agent has a sufficient information and memory regarding the samples. Without assuming such an informational assumption, this paper attempts to provide an alternative explanation for why the prior belief of the agent can be assumed to coincide with the correct one, by showing that

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[1] For the hypothesis tests in the philosophical literature, see Sober (2008, Ch.1). Okasha (2013) and Lo and Zhang (2021) derive the Bayesian behavioral hypothesis (updating of prior and utility maximization based on posterior) as a property of long-run equilibria of evolutionary process in which non-Bayesian acts are permitted. Zhang (2013) shows that non-Bayesian behavioral hypothesis is derived when individual's risk aversion is incorporated into the process. In this paper, the Bayesian behavioral hypothesis is major premise; we do not introduce the non-Bayesian hypotheses in this paper.

[2] For statistical inference theory, see DeGroot and Schervish (2012, Ch 7). In particular, for the theories of determiniation of the prior beliefs, see Berger, Bernardo and Sun (2015) and the references. By way of the statistical inference, it is possible that a testing agent eventually knows the true probability with some awareness.

the testing agents with the correct prior belief are naturally selected in an evolutional test model.[3]

We first demonstrate that a simple Bayesian hypothesis test scheme has the good property whereby the probability of the testing agent to obtain the optimal outcome is maximized *only* if the prior belief of the agent coincides with the correct one, when the agent selects the Bayesian-optimal test procedure with respect to his or her (own) prior belief, although a successful decision maker need not have been endowed with the correct prior belief in general.[4] If the fitness value is determined by the test outcome in an evolutional setting, we can re-state this result as follows: The simple Bayesian hypothesis test scheme has a good property whereby a testing agent obtains the *evolutionally* optimal outcome *only* if the prior belief of the agent coincides with the correct one (Proposition 2).

More precisely, the test scheme has this property under an additional assumption for the sample distribution, which is not introduced in the test theory. However, we show that the additional assumption is necessary for the test scheme to have this property by constructing an example of a test in which the property does not hold if the assumption is removed.

Second, we apply the aforementioned result to an evolutional test model in which the Bayesian test is conducted repeatedly in parallel by many testing agents. We assume that the agents have the diverse initial prior beliefs, which can be derived from a

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[3] As an explanation of such an assertion, our approach is closely related to the pragmatic explanation why an expert acts as if the expert knows the underlying information that is a crucial factor for determining the outcome of the act. See Milton Friedman (1953, Section III) for example. In the literature of evolutionary epistemology, Downes (2000) and Boudry and Vlerick (2014) consider the possibility that the true beliefs are naturally selected.

[4] In the horse races, the single win scheme has the property, but the double win scheme does not, because not only the full information for the true arrival order but also the full information for the other betters' betting behaviors is required to obtain the best result in the double win scheme.

hypothetical experiment with the agents who base their decision on Laplace's principle of insufficient reason. Then, only the testing agents with the correct belief survive through the Malthusian competition (Proposition 3).

Because a Bayesian hypothesis test can be regarded as a one-shot Bayesian hypothesis test that appears after many generations have occurred, the latter result (Proposition 3) provides an explanation for why the prior belief of the agent can be assumed to coincide with the correct one in a Bayesian hypothesis test, as if the agent knows the true probability assigned by the nature.[5]

The next section introduces the Bayesian hypothesis test, and Section 3 introduces the Bayesian-optimal test procedure and shows its well-known characterization (Proposition 1). Section 4 evaluates the Bayesian-optimal test procedures objectively and states the main result of this paper (Proposition 2). The dynamic process is introduced, and Proposition 3 is stated in Section 5.

2. The Bayesian hypothesis test

This section introduces the (one-shot) Bayesian hypothesis test model with a testing agent. As in a standard textbook (Lesaffre and Lawson, 2012), we formulate the Bayesian test as a diagnostic test in which the testing agent (a medical doctor) conducts a test for the medical treatments of a patient randomly selected from a large population. We make the following assumptions:

$\mathbf{A}_1$: The patient contracts tuberculosis (TB), or the patient suffers from a normal cold (NC). Nature determines one of the two events, TB or NC, exclusively for all patients in a city. Let p^* be the true probability of TB (the true prevalence rate), and we assume that $1 > p^* > 0$.

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[5] Even though the agents know the true probability without awareness, this result provides a reason why the common prior assumption in the game theory holds for the agents.

$\mathbf{A_2}$: For a medical doctor, the patient is selected randomly from a large population of patients in the city. The medical doctor does not know the true probability, p^*. But the medical doctor has a subjective probability that the patient is TB, which we call a prior belief. Let $P \equiv (0, 1)$ be the set of all possible prior beliefs. The two simple hypotheses are given as follows:

the null hypothesis (H_0): The patient suffers from a normal cold, NC.

the alternative hypothesis (H_1): The patient contracts tuberculosis, TB.

$\mathbf{A_3}$: A medical doctor examines a patient by a TB check-up kit. The check-up kit indicates a real number $x \in X \equiv \mathbb{R}$, which is assumed to be the sample derived from the random device of which the probability distribution is $f(x \mid \theta_0)$ in case the patient suffers from a NC. (This means that the sample size is one.) Otherwise, we assume that the check-up kit indicates a real number $x \in X$, which is assumed to be the sample derived from the random device of which probability destribution is $f(x \mid \theta_1)$. We assume that $f(x \mid \theta_0)$ and $f(x \mid \theta_1)$ are differentiable functions of $x \in X$, and that $f(x \mid \theta_0) > 0$ and $f(x \mid \theta_1) > 0$ for all $x \in X$. We define the *likelihood ratio function* $L(x)$ by $L(x) \equiv f(x \mid \theta_1)/f(x \mid \theta_0)$, and we assume that $L(x)$ is a *monotone* function of $x \in X$. Namely, we assume the following:

$$L'(x) > 0 \text{ for all } x \in X, \text{ or } L'(x) < 0 \text{ for all } x \in X. \tag{1}$$

The monotonicity assumption is a standard assumption in the test theory. See DeGroot and Schervish (2012, Ch.9, Definition 9.3.2).

$\mathbf{A_4}$: The medical doctor selects one of the two acts, "Do not reject H_0 and prescribe streptomycin" or "Reject H_0 and prescribe antipyretics". The former act is denoted by d_0 and the latter act is denoted by d_1. The selection of an act can be determined conditioned on the number in X given by $\mathbf{A_3}$, and the selection is represented by a function $\delta : X \to \{ d_0, d_1 \}$ such that $\delta^{-1}(d_0) \in \mathfrak{B}_X$, where $\mathfrak{B}_X$ is the σ-field of Borel subsets in X. The (completed) Lebesgue measure on $\mathfrak{B}_X$ is denoted by μ. In the following, the selection of

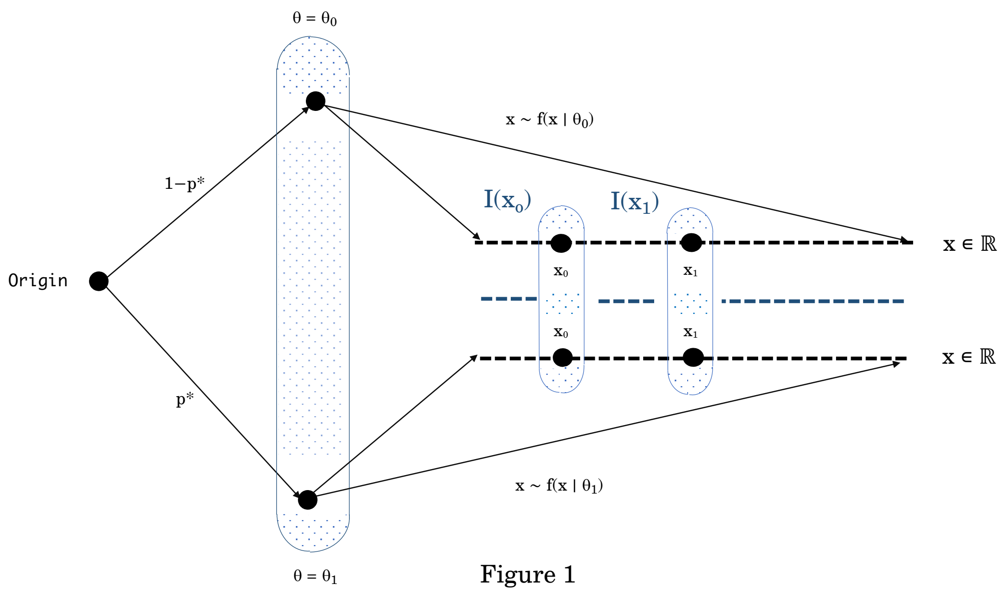

Figure 1

act δ is called a test procedure, and the set of all test procedures is denoted by Δ. **A**$_5$: If the event selected by nature at **A**$_1$ is consistent with the act $\delta(x)$ selected at **A**$_4$, then there is no error, otherwise an error occurred. Concretely, there are two types of errors:

An error is happened if and only if $\delta(x) = d_1$, when $\theta = \theta_0$;

An error is happened if and only if $\delta(x) = d_0$, when $\theta = \theta_1$.

The former errors are called *Type I errors*, and the latter errors are called *Type II errors*. Although the doctor remembers the act $\delta(x)$ at **A**$_4$, the doctor is not informed of the true state, $(\theta = \theta_0)$ or $(\theta = \theta_1)$, and the doctor can not guess if an error occurred or not, as shown in Figure 2 below: [6]

[6] This assumption implies that the Bayesian updating is conducted within the one-shot test and that the agents cannot use the genetically retained updating process. If we assume the genetically retained updating or the existence of a central authority that attempts to infer the (true) posterior probabilities, it is not difficult to derive the (correct) Bayesian test procedure

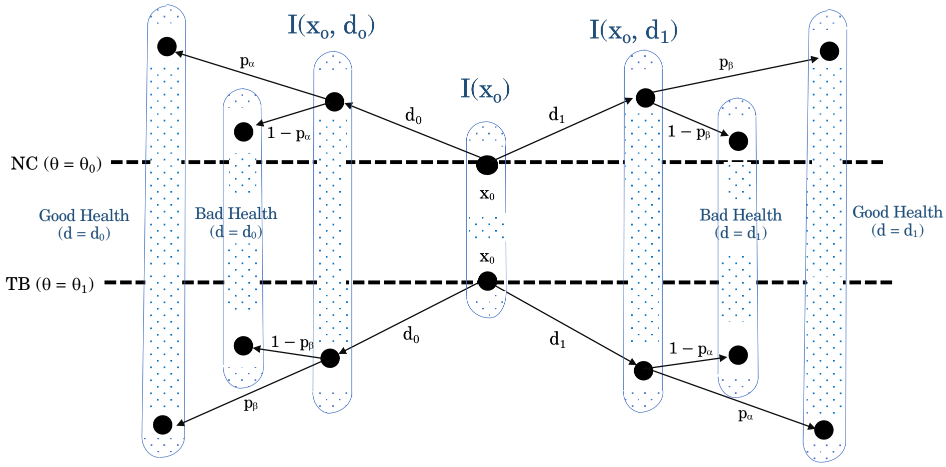

Figure 2

A₆: After the medical treatment, nature randomly determines the final outcome, "The patient is healthy" or "The patient is not healthy".[7] In case of no error, the probability of "The patient is healthy" is denoted by p_α. In case of errors, the probability of "The patient is healthy" is denoted by p_β. We assume that $1 > p_\alpha > p_\beta > 0$.

3. The Bayesian-optimal test procedures

In the test model constructed in the previous section, this section introduces the Bayesian

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as shown in Footnote 9 in Section 3 of this paper. Moreover, in a one-shot test, if the agent can derive a large sample, the agent inferes the true hypothesis, making use of the Bayesian updating. Because the sample size is always equal to one in this paper, it is very difficult for the agent to take such an inference strategy.

[7] In a standard textbook as DeGroot and Schervish (2012, Ch 9, Section 9.8), only the cost of a treatment (or decision) is introduced in order to evaluate the test procedure. In this paper, the probabilities for good health mean the benefit of a treatment. There is no essential defference, because minimizing the cost is equivalent to maximizing the benefit as shown in Section 3 of this paper. We can derive almost the same results, even if we do not assume that the probabilities for the two types errors are the same.

behavioral assumption for the agent and the Bayesian-optimal test procedure. Namely, for a given prior belief, using the realization of the sample and the updating rule of the belief, the testing agent computes the posterior probabilities, and the agent selects the test procedure to maximize the probability that the patient is healthy after the medical treatment.

Concretely, the agent's prior belief for the occurrence of the event that a patient has TB ($\theta = \theta_1$) is specified by a *subjective probability* q in P. For a prior belief $q \in$ P, let $P(\theta_i | x: q)$ be the posterior probability of ($\theta = \theta_i$) after receiving signal $x \in$ X for i = 0, 1.[8] Using the Bayes theorem, we have the following well-known lemma:

Lemma 1: $P(\theta_1|x{:}q) = \frac{q\cdot f(x|\theta_1)}{q\cdot f(x|\theta_1)+(1-q)\cdot f(x|\theta_0)}$ and $P(\theta_0|x{:}q) = \frac{(1-q)\cdot f(x|\theta_0)}{q\cdot f(x|\theta_1)+(1-q)\cdot f(x|\theta_0)}$.

	d_0	d_1
θ_0	p_α	p_β
θ_1	p_β	p_α

Table 1.

For a prior belief $q \in$ P, let $h(d_i | x: q)$ be the posterior probability of "The patient is healthy" in case of the medical treatment d_i after receiving signal $x \in$ X for i = 0, 1. The following holds by the definition of $P(\theta_i | x{:}q)$ and $\mathbf{A_6}$:

$$h(d_1 | x{:}q)=P(\theta_0 | x{:}q)\cdot p_\beta +P(\theta_1 | x{:}q)\cdot p_\alpha \text{ and } h(d_0 | x{:}q)=P(\theta_0 | x{:}q)\cdot p_\alpha+P(\theta_1 | x{:}q)\cdot p_\beta. \quad (2)$$

As a direct consequence of Lemma 1 and Equation (2), we may posit another lemma:

Lemma 2: (**i**) $h(d_1 | x{:}\ q) = \frac{(1-q)\cdot p_\beta\cdot f(x | \theta_0)+q\cdot p_\alpha\cdot f(x | \theta_1)}{q\cdot f(x | \theta_1)+(1-q)\cdot f(x | \theta_0)}$ and $h(d_0 | x{:}\ q) = \frac{(1-q)\cdot p_\alpha\cdot f(x | \theta_0)+q\cdot p_\beta\cdot f(x | \theta_1)}{q\cdot f(x | \theta_1)+(1-q)\cdot f(x | \theta_0)}$.

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[8] Setting $X_n = (x-(1/n), x+(1/n))$, we define $P(\theta_1 | x{:}q)$ by $P(\theta_1 | x{:}q) = \lim_n q\cdot P(X_n | \theta_1)/P(X_n)$, where $P(X_n | \theta_1) = \int_{X_n} f(x | \theta_1)d\mu$ and $P(X_n) = \int_{X_n} [q\cdot f(x | \theta_1)+(1-q)\cdot f(x | \theta_0)]d\mu$. We can define $P(\theta_0 | x{:}q)$ by almost the same manner.

(**ii**) For all $q \in P$, $h(d_1 \mid x: q)$ and $h(d_0 \mid x: q)$ are continuous for $x \in X$.

For prior belief $q \in P$, $h(\delta(x) \mid x: q)$ can be recognized as the probability of "The patient is healthy" of a test procedure $\delta \in \Delta$ after receiving signal x. The testing agent attempts to maximize the probabilities $h(\delta(x) \mid x: q)$ $(x \in X)$ by selecting the suitable test procedure $\delta \in \Delta$. We define the optimal test procedure δ as follows: For a given prior belief $q \in P$, a test procedure $\delta \in \Delta$ is defined to be a *Bayesian-optimal test procedure with respect to q* if and only if:

$$h(\delta(x) \mid x: q) = \mathbf{max}[h(d_1 \mid x: q), h(d_0 \mid x: q)] \text{ for almost all } x \in X,^{9} \quad (3)$$

which means that the probability of "The patient is healthy", $h(\delta(x) \mid x: q)$ is maximal in the set $\{ h(d_1 \mid x: q), h(d_0 \mid x: q) \}$ for almost all $x \in X$. Equation (3) above specifies the property of a test procedure only, and there is no specification for a relative advantage over the other test procedures.

For the next Proposition, we require some definitions and Lemmas. For prior belief $q \in P$, let $G(\delta: q)$ be the induced probability of "The patient is healthy" if a testing agent selects a test procedure $\delta \in \Delta$. Then, according to $\mathbf{A_5}$ and $\mathbf{A_6}$ the following holds:

$$G(\delta: q) = (1-q)\cdot(1-\alpha(\delta))\cdot p_\alpha+(1-q)\cdot\alpha(\delta)\cdot p_\beta+q\cdot(1-\beta(\delta))\cdot p_\alpha+q\cdot\beta(\delta)\cdot p_\beta, \quad (4)$$

where $\alpha(\delta)$ is the probability of Type I errors of δ defined by $\alpha(\delta) = \int_{\delta^{-1}(d_1)} f(x|\theta_0)d\mu$ and $\beta(\delta)$ is the probability of Type II errors of δ defined by $\beta(\delta) = \int_{\delta^{-1}(d_0)} f(x|\theta_1)d\mu$.[10]

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[9] When $A \in \mathfrak{B}_X$ and $\mu(A) > 0$, a statement of x holds for *almost* all $x \in A$ if and only if there exists $B \subset A$ such that: (i) $B \in \mathfrak{B}_X$, (ii) $\mu(B) = \mu(A)$, (iii) The statement of x holds for *all* $x \in B$. The function $w(x) \equiv \mathbf{max}[h(d_1 \mid x: q), h(d_0 \mid x: q)]$ is Borel measurable, because *w(x)* is continuous on X by Lemma 2(ii) and Royden (1988, Problems 4 and 5 in Page 34, and Problem 44 in Page 49). Suppose that there exists a statistician who can collect the information from the many agents such as (x, d, h), where x is a realization of the sample, d is the decision, and h is the final health condition of the patient. If the statistician gets sufficient information, then the statistician can infer the probabilities $h(d_1 \mid x: p^*)$ and $h(d_0 \mid x: p^*)$, where p^* is the true probability, and then derives the Bayesian-optimal test procedure with respect to p^* without knowing the true probabilities $[f(x \mid \theta_0), f(x \mid \theta_1), p_\alpha, p_\beta, p^*]$.

[10] The numerical value of $G(\delta: q)$ can be interpreted as the value of the expected utility defined by $G(\delta: q)\cdot$(utility of a healthy state)+$(1-G(\delta: q))\cdot$(utility of a not healthy state), because the

Lemma 3: (**i**) For all $q \in P$ and all $\delta^1, \delta^2 \in \Delta_q$, it holds that $G(\delta^1: q) = G(\delta^2: q)$. (**ii**) It holds that $G(\delta: q) = \int_X h(\delta(x) \mid x: q)d\lambda_q$, where λ_q is the measure of $(X, \mathfrak{B}_X)$ defined by $\lambda_q(B) = \int_B [q \cdot f(x \mid \theta_1)+(1-q)\cdot f(x \mid \theta_0)]d\mu$ for all $B \in \mathfrak{B}_X$.

For prior belief $q \in P$, a test procedure $\delta^* \in \Delta$ is defined to be a *subjectively optimal test procedure with respect to q* if and only if:

$$G(\delta^*: q) \geq G(\delta: q) \text{ for all } \delta \in \Delta. \tag{5}$$

Statement (5) means that the test procedure δ^* maximizes the probability of "The patient is healthy", which is computed from belief q. According to (4), it holds that $G(\delta: q) = p_\alpha - [(1-q)\cdot(p_\alpha-p_\beta)\cdot\alpha(\delta)+q\cdot(p_\alpha-p_\beta)\cdot \beta(\delta)]$, which implies that the maximization of $G(\delta: q)$ is equivalent to the minimization of the cost, $C(\delta: q) \equiv (1-q)\cdot(p_\alpha-p_\beta)\cdot\alpha(\delta)+q\cdot(p_\alpha-p_\beta)\cdot\beta(\delta)$, by which we can define the *likelihood ratio test procedure* in the standard way as in DeGroot and Schervish (2012, Ch.9, Section 9.2, Corollary 9.2.1). Concretely, using the weights $(1-q)\cdot(p_\alpha-p_\beta)$ and $q\cdot(p_\alpha-p_\beta)$ for the probabilities of errors $\alpha(\delta)$ and $\beta(\delta)$, we can define the *critical likelihood ratio at* $q \in P$ by $D(q) = (1-q)\cdot(p_\alpha-p_\beta)/q\cdot(p_\alpha-p_\beta) = (1-q)/q$. Then, the test procedure $\delta^* \in \Delta$ is defined to be a *likelihood ratio test procedure with respect to q* if and only if:

$$\delta^*(x) = d_1 \text{ for almost all } x \in C^1(q);\ \ \delta^*(x) = d_0 \text{ for almost all } x \in C^0(q), \tag{6}$$

where $C^1(q) = \{ x \in X: f(x \mid \theta_1)/f(x \mid \theta_0) > D(q) \}$ and $C^0(q) = \{ x \in X: f(x \mid \theta_1)/f(x \mid \theta_0) < D(q) \}$. Let Δ_q be the set of all likelihood ratio test procedures with respect to $q \in P$.

Lemma 4: For any prior belief $q \in P$, the following assertions hold: (**i**) $\Delta_q \neq \emptyset$. (**ii**) $\{ x \in X: f(x \mid \theta_1)/f(x \mid \theta_0) > (1-q)/q \} = \{ x \in X: h(d_1 \mid x: q) > h(d_0 \mid x: q) \}$, $\{ x \in X: f(x \mid \theta_1)/f(x \mid \theta_0) < (1-q)/q \} = \{ x \in X: h(d_1 \mid x: q) < h(d_0 \mid x: q) \}$, and $\{ x \in X: f(x \mid \theta_1)/f(x \mid \theta_0) = (1-q)/q \} = \{ x \in X: h(d_1 \mid x: q) = h(d_0 \mid x: q) \}$. (**iii**) If $\delta^1 \in \Delta_q$ and $\delta^2 \in \Delta$, then $h(\delta^1(x) \mid x: q) \geq h(\delta^2(x) \mid x: q)$ for almost all $x \in X$. (**iv**) If $\delta^1 \in \Delta_q$ and $\delta^2 \in \Delta_q$, then $h(\delta^1(x) \mid x: q) = h(\delta^2(x) \mid x: q)$ for almost all $x \in X$.

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utility value coincides with $G(\delta:q)$, if we set (utility of a healthy state) = 1 and (utility of a not healthy state) = 0.

($\mathbf{v}$) Suppose that $\delta^1 \in \Delta_q$. If $\delta^2 \in \Delta$ satisfies $\delta^2 \notin \Delta_q$, then there exists a compact subset A in X such that $\mu(A) > 0$ and $h(\delta^1(x) | x: q) > h(\delta^2(x) | x: q)$ for all $x \in A$. (**vi**) If $\delta^1 \in \Delta_q$ and $\delta^2 \notin \Delta_q$, then $G(\delta^1: q) > G(\delta^2: q)$.

We have the following well-known proposition[11]:

Proposition 1(Existence and characterization of the Bayesian-optimal test procedures): Select any $q \in P$ and let q be the subjective belief of a testing agent.

(**i**) There exists a Bayesian-optimal test procedure with respect to q.

(**ii**) The following three statements are mutually equivalent:

> (**a**) The test procedure $\delta \in \Delta$ is Bayesian-optimal with respect to q.
>
> (**b**) The test procedure $\delta \in \Delta$ is subjectively optimal with respect to q.
>
> (**c**) The test procedure $\delta \in \Delta$ is a likelihood ratio test procedure with respect to q.

(**iii**) Let δ^* be a Bayesian-optimal test procedure with respect to q. If a test procedure δ is not a Bayesian-optimal test procedure with respect to q, then $G(\delta^*: q) > G(\delta: q)$.

Proof of Proposition 1: (**ii**) First, we prove the assertion (ii) of Proposition 1. [(**a**) ⇒ (**c**)]: Because $\Delta_q \neq \emptyset$ by Lemma 4(i), we can assume that there exists $\delta^* \in \Delta_q$. If $\delta^0 \in \Delta$ is not a likelihood ratio test procedure with respect to q, then it holds by Lemma 4(v) that there exists a compact subset A in X such that $\mu(A) > 0$ and $h(\delta^*(x) | x : q) > h(\delta^0(x) | x : q)$ for all $x \in A$, which implies that $\delta^0 \in \Delta$ is not a Bayesian-optimal test procedure with respect to q. Taking the contraposition of this, we then find that a Bayesian optimal test procedure with respect to q is a likelihood ratio test procedure with respect to q. [(**c**) ⇒ (**b**)]: Let δ^* be a likelihood ratio test procedure with respect to q. Suppose that δ^* is not subjectively optimal. It holds according to (5) that there exists $\delta^0 \in \Delta$ and a compact subset A in X such that $\mu(A) > 0$ and $h(\delta^0(x) | x : q) > h(\delta^*(x) | x : q)$ for all $x \in A$. However, it holds by Lemma

[11] Specifically, Proposition 1(ii)[(b)⇔(c)] is a re-statement of Neyman-Pearson lemma as in DeGroot and Schervish (2012, Ch.9, Theorem 9.2.1) and Lehmann and Romano (2005, Ch.3). For Proposition 1(ii)[(a)⇔(c)], see DeGroot and Schervish (2012, Ch.9, Section 9.8).

4(iii) that $h(\delta^*(x) \mid x : q) \geq h(\delta^0(x) \mid x : q)$ for almost all $x \in A$. This is a contradiction, and we have that δ^* is subjectively optimal. [(**b**) $\Rightarrow$ (**c**)]: Because $\Delta_q \neq \emptyset$ by Lemma 4(i), we can assume that there exists $\delta^* \in \Delta_q$. If $\delta \in \Delta$ is not a likelihood ratio test procedure with respect to q, then it holds by Lemma 4(vi) that $G(\delta^*: q) > G(\delta: q)$, which implies that δ is not subjectively optimal with respect to q. Taking the contraposition of this, we find that a subjectively optimal test procedure with respect to q is a likelihood ratio test procedure with respect to q. [(**c**) $\Rightarrow$ (**a**)]: Let δ^* be a likelihood ratio test procedure with respect to q. For any $\delta \in \Delta$, it holds by Lemma 4(iii) that $h(\delta^*(x) \mid x : q) \geq h(\delta(x) \mid x : q)$ for almost all $x \in X$, which implies that δ^* is a Bayesian-optimal test procedure with respect to q. (**iii**) Proposition 1(iii) is a direct consequence of Lemma 4(vi) and Proposition 1(ii)[(**a**) $\Leftrightarrow$ (**c**)]. (**i**) Proposition 1(i) is a direct consequence of Proposition 1(ii) and Lemma 4(i). □

For any prior belief $q \in P$, the existence of a Bayesian-optimal test procedure with respect to q is ensured by Proposition 1(i). The optimality concept of (**a**) is defined by the best responses to possible signals, whereas the optimality concept of (**b**) is defined by the comparison of the values directly on the set of all test procedures. In terms of the game theory, the former corresponds to the optimality in the behavioral strategies of extenstive form games; the latter corresponds to the optimality in the pure strategies of strategic form games. Proposition 1(ii)[(**a**) $\Leftrightarrow$ (**b**)] shows that these optimality concepts are equivalent in the Bayesian test scheme. Proposition 1(iii) strengthens the optimality condition by showing that the value of G in the Bayesian-optimal test procedure is *strictly* greater than the values of G in the non-optimal test procedures.

Example 1: We assume that the conditional probability distributions, $f(x \mid \theta_0)$ and $f(x \mid \theta_1)$ in $\mathbf{A}_3$ are given by $f(x \mid \theta_0) = A \cdot exp[-(½) \cdot x^2]$ and $f(x \mid \theta_1) = A \cdot exp[-(½) \cdot (x-1)^2]$ for all $x \in X$, where $A = (2\pi)^{-½}$ and $exp(y) = e^y$ for all $y \in X$.

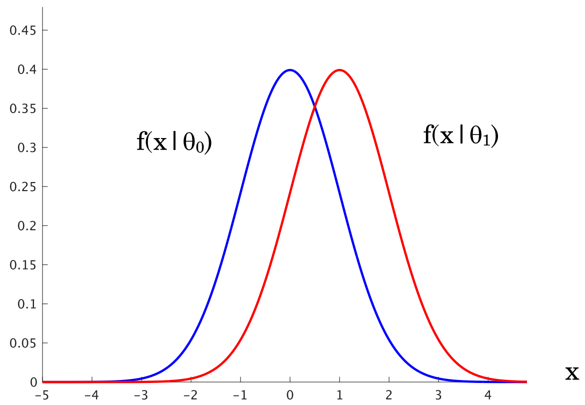

Figure 3

The critical likelihood ratio at $q \in$ P is given by D(q) = (1−q)/q. Then it holds by (4) that the likelihood ratio function L(x) is given by L(x) = *exp*(x−(½)). When q = 0.2, D(0.2) = 4, and L^{-1}(D(0.2)) = *log*(4)+0.5 ≒ 1.886.

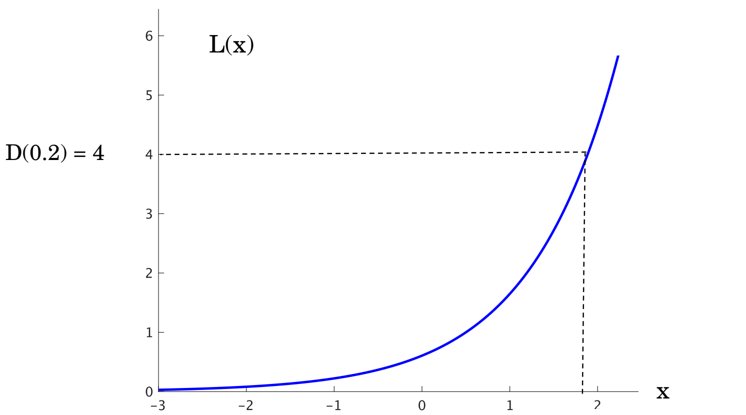

Figure 4

Example 2: We assume that the conditional probability distributions, $f(x \mid \theta_0)$ and $f(x \mid \theta_1)$ in $\mathbf{A}_3$ are given by $f(x \mid \theta_0) = A \cdot exp[-(½) \cdot x^2]$ and $f(x \mid \theta_1) = A \cdot exp[-(½) \cdot x^2] \cdot g(x)$ for all $x \in X$, where $A = (2\pi)^{-½}$ and $g(x) = 2 \cdot e^x/(e^x+1)$ for all $x \in X$.

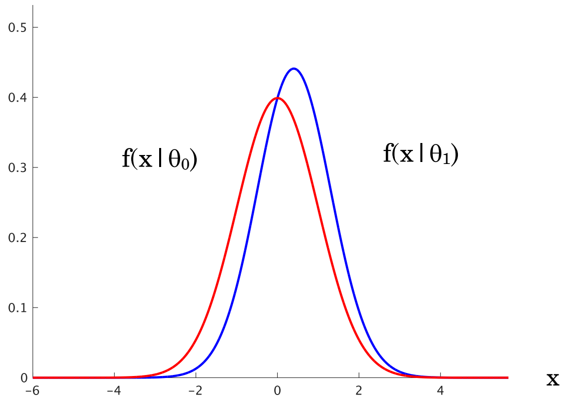

Figure 5

The likelihood ratio function L(x) is given by L(x) = $2 \cdot e^x/(e^x+1)$.

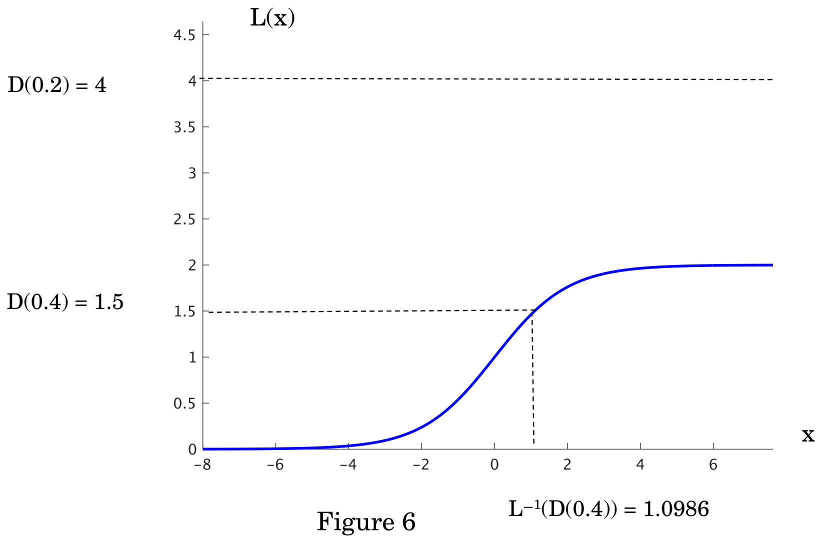

Figure 6

When q = 0.2, D(0.2) = 4, $C^1(0.2) = \{ x \in X\text{: } L(x) > 4 \} = \emptyset$ and $C^0(0.2) = \{ x \in X\text{: } L(x) < 4 \}$ = X. Hence a typical likelihood ratio test procedure δ^* in $\Delta_{0.2}$ is given by $\delta^*(x) = d_0$ for all $x \in X$. When q = 0.4, D(0.4) = 1.5 and $L^{-1}(D(0.4)) = log(3) ≒ 1.0986$. Hence a typical likelihood ratio test procedure δ^* in $\Delta_{0.4}$ is given by $\delta^*(x) = d_1$ for all $x \geq 1.0986$; $\delta^*(x) = d_0$ for all $x < 1.0986$.

4. The test strategy concept and the evolutionally optimal test strategy

It holds by Proposition 1 that if a testing agent attempts to maximize the subjective probability of the event that the patient is healthy, then the agent selects a likelihood ratio test procedure in Δ_q, where q is the prior belief of the agent. We assume the following:

A$_7$: The fitness value coincides with the (objective) probability of the event that the patient is healthy for each testing agent.

Suppose that $p^* \in P$ is the true probability (the true prevalence rate). If a testing agent selects a test procedure $\delta \in \Delta$, the fitness value of the agent denoted by $G(\delta: p^*)$ can be defined by **A$_7$** and (4) as follows:

$$G(\delta: p^*) = (1-p^*)\cdot(1-\alpha(\delta))\cdot p_\alpha+(1-p^*)\cdot\alpha(\delta)\cdot p_\beta + p^*\cdot(1-\beta(\delta))\cdot p_\alpha+p^*\cdot\beta(\delta)\cdot p_\beta\,. \qquad (7)$$

Lemma 5: Set $\alpha^*(q) = \int_{C^1(q)} f(x|\theta_0)d\mu$ and $\beta^*(q) = \int_{C^0(q)} f(x|\theta_1)d\mu$. It holds that $\alpha^*(q) = \alpha(\delta)$ and $\beta^*(q) = \beta(\delta)$ for all $q \in P$ and all $\delta \in \Delta_q$.

We define a function, $F(\cdot, p^*)$ on P as follows:

$$F(q: p^*) = (1-p^*)\cdot(1-\alpha^*(q))\cdot p_\alpha+(1-p^*)\cdot\alpha^*(q)\cdot p_\beta$$
$$+p^*\cdot(1-\beta^*(q))\cdot p_\alpha+p^*\cdot\beta^*(q)\cdot p_\beta \text{ for all } q \in P. \qquad (8)$$

Lemma 6: (**i**) $F(q: p^*) = G(\delta: p^*)$ for all $q \in P$ and all $\delta \in \Delta_q$. (**ii**) $F(q: p^*)$ is continuous with respect to $q \in P$.

The function $F(\cdot, p^*)$ need not to be an injection on P. The value of $F(q: p^*)$ coincides with the fitness value, when the testing agent takes the test procedure in Δ_q. Later in this paper, Δ_q is called a *test strategy corresponding to q*. Test strategy Δ_q is defined to be an *evolutionally optimal strategy with respect to p^** if and only if:

$$F(q: p^*) > F(r: p^*) \text{ for all } r \in P \text{ such that } r \neq q. \qquad (9)$$

Let us consider the two examples again:

Example 1.(continued) Set $p^* = 0.4$. We can draw the graph of the function $F(q: 0.4)$ for this example, using the argument for deriving the procedure δ^* in $\Delta_{0.2}$. Therefore, $\Delta_{0.4}$ is an evolutionally optimal test strategy with respect to p* = 0.4.

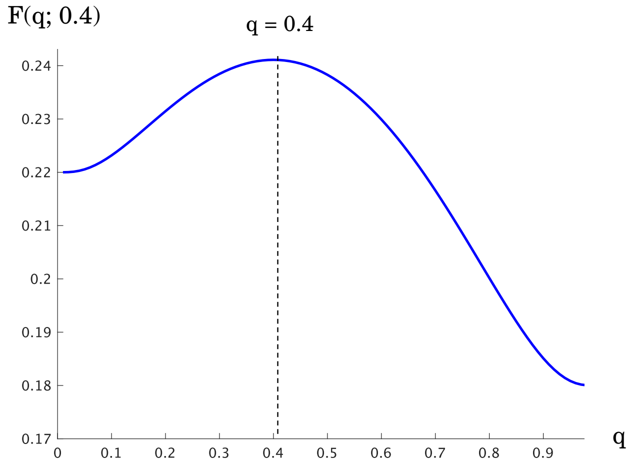

Figure 7

Example 2.(continued) Set $p^* = 0.2$, $p_\alpha = 0.3$, $p_\beta = 0.1$. Following the arguments for deriving the procedure δ^* in $\Delta_{0.2}$ and (7), we have that $\Delta_q = \Delta_r$ for all $q, r < d^{-1}(2) = 1/3$, and that:

$$F(q: 0.2) = p^* \cdot p_\beta + (1-p^*) \cdot p_\alpha = 0.26 \qquad \text{for all } q < 1/3$$

$$= 0.3 - [0.16\alpha^*(q) + 0.04\beta^*(q)] \quad \text{for all } q \geq 1/3.$$

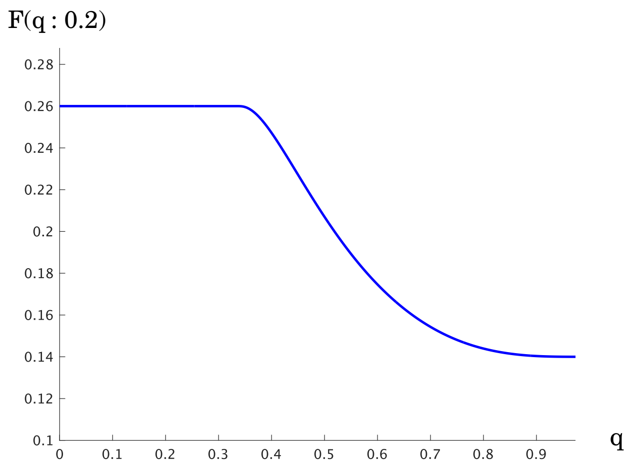

Figure 8

Therefore, there is no evolutionally optimal test strategy with respect to p^* in Example 2.

To ensure the existence of an evolutionally optimal test strategy with respect to the true probability (the true prevalence rate), p*, we additionally assume the following condition:

A_8: For any $y \in \mathbb{R}_{++}$, there exists some $x \in \mathbb{R}$ such that $y = f(x \mid \theta_1)/f(x \mid \theta_0)$.

Because **A_8** means the range of $L(x) \equiv f(x \mid \theta_1)/f(x \mid \theta_0)$ coincides with the full set $\mathbb{R}_{++}$, we call **A_8** the full range condition of the likelihood ratio. Under the full range condition of the likelihood ratio, we obtain the following lemma:

Lemma 7: $\Delta_q \cap \Delta_r = \emptyset$ for all $q, r \in P$ such that $q \neq r$.

As a main result of this paper, we make the following proposition:

Proposition 2(Existence and characterization of the evolutionally optimal test strategy): Under all the assumptions stated above, the following assertions hold:

(**i**) There exists an evolutionally optimal test strategy with respect to p*.

(**ii**) Test strategy Δ_q corresponding to a prior belief q is evolutionally optimal with respect to p* if and only if the prior belief q coincides with p*.

(**iii**) Let Δ_q be an evolutionally optimal test strategy with respect to p*. Then, it holds that $G(\delta^*: p^*) > G(\delta: p^*)$ for all $\delta^* \in \Delta_q$ and all $\delta \notin \Delta_q$.

Proof of Proposition 2: (**ii**) Suppose that Δ_q is an evolutionally optimal test strategy with respect to p* and that $q \neq p^*$ holds. We have by (9) that $F(q: p^*) > F(p^*: p^*)$. If $\delta \in \Delta_q$, it holds by Lemma 7 that $\delta \notin \Delta_{p^*}$. Fix any $\delta^* \in \Delta_{p^*}$. It holds by Proposition 1(iii) that $G(\delta^*: p^*) > G(\delta: p^*)$, which implies that $F(p^*: p^*) > F(q: p^*)$. This is a contradiction. Therefore, we have that $q = p^*$. Conversely, suppose that $q = p^*$. Let $r \in P$ be a prior belief such that $r \neq q$. Suppose that $\delta \in \Delta_r$ holds. Then, it holds by Lemma 7 that $\delta \notin \Delta_{p^*}$. Fix any $\delta^* \in \Delta_{p^*}$. We have by Proposition 1(iii) that $G(\delta^*: p^*) > G(\delta: p^*)$, which implies that $F(q: p^*) > F(r: p^*)$. (**i**) Proposition 2(i) holds by Proposition 2(ii) and Lemma 4(i). (**iii**) Suppose that $\delta^* \in \Delta_q$ and $\delta \notin \Delta_q$. It holds by Proposition 2(ii) that $\delta^* \in \Delta_{p^*}$ and $\delta \notin \Delta_{p^*}$. We have by

Proposition 1(iii) that $G(\delta^*: p^*) > G(\delta: p^*)$. □

For a true probability $p^* \in P$, Proposition 2(i) ensures that there exists an evolutionally optimal test strategy with respect to p^*. Proposition 2(ii) implies that the unique evolutionally optimal test strategy is Δ_{p^*}. Proposition 2(iii) implies that any test procedure in the evolutionally optimality strategy Δ_{p^*} dominates all of the test procedures in $(\Delta_{p^*})^c \equiv \Delta/\Delta_{p^*}$, which implies that Δ_{p^*} dominates any mutation strategies.

5. A dynamic process for selecting the testing agents with the correct prior beliefs

This section introduces a dynamic test model in which the Bayesian test is conducted repeatedly in parallel by many medical doctors (testing agents), and it is shown that only the medical doctors with the correct prior belief survive through the Malthusian competition. We additionally assume the following conditions:

A_9 (Time structure and population measures): Time is discrete and infinite, and it is denoted by $t = 1, 2, \cdots$. Without specifying the set of medical doctors, the population measure of the doctors at t is represented by the measure π_t on P for $t = 1, 2, \cdots$.

A_{10} (Initial condition): At the beginning of time 1, we assume that there is a continuum of doctors $N_1 = (0, 1)$, and that all the doctors in N_1 form the initial priors based on Laplace's principle of insufficient reason. Concretely, we consider a hypothetical experiment in which each doctor determines the initial prior by drawing a lottery defined by the uniform probability distribution on the set of all possible priors. Then, all the doctors draw the lottery independently and simultaneously. As the result of the experiment, there is a population measure π_1 on P,[12] which coincides with the Lebesgue μ on P. The density function h_1 of π_1 is defined as follows:

$$h_1(q) = 1 \text{ for all } q \in P. \tag{10}$$

[12] Namely, we apply Laplace's principle for an experiment with a double infinity of events and doctors.

$\mathbf{A}_{11}$ (**Transition rule**): At the beginning of period t, for each medical doctor who is active at t, a patient is selected randomly from a large population of patients. The patient is tested, and the doctor provides the medical treatment depending on the test result (signal) specified by $\mathbf{A}_3$. If the patient's final health condition is good, the doctor can leave an offspring, otherwise the doctor cannot leave an offspring. At the end of period t, all the doctors enter retirement, and the offsprings will appear and become active doctors in the next period t+1. We assume that the offspring's prior belief coincides with the parent's prior belief. Moreover, we assume that the measure of each type of offspring is increased by the natural rate of population growth $\gamma_t > 0$. Formally, we assume the following:

$$\lim_{k\to+\infty} \frac{\pi_{t+1}(\mathrm{B(q,k)})}{\pi_t(\mathrm{B(q,k)})} = \mathrm{F(q;\ p^*)}\cdot(1+\gamma_t) \text{ for all } q \in \mathrm{P} \text{ and all } t = 1, 2, 3, \cdots, \tag{11}$$

where $\mathrm{B(q, k)} \equiv \{ \mathrm{r} \in \mathrm{P}: | \mathrm{r-q} | < (1/\mathrm{k}) \}$ for all $q \in \mathrm{P}$ and all k = 1, 2, 3, ⋯ . A sequence of measures $\{ \pi_t \}_{t=1}^{+\infty}$ satisfying (11) is called a *dynamic process*.

Lemma 8: Let $\{ \pi_t \}_{t=1}^{+\infty}$ be a dynamic process, and let h_1 be the Radon-Nikodým density function of π_1. Then all population measures π_t (t = 2, 3, ⋯) are absolutely continuous with respect to the Lebesgue μ on P, and it holds that

$$\frac{h_{t+1}(\mathrm{q})}{h_t(\mathrm{q})} = \mathrm{F(q;\ p^*)}\cdot(1+\gamma_t) \text{ for all } q \in \mathrm{P} \text{ and all } t = 1, 2, \cdots,$$

where h_t be the density function of π_t for t = 2, 3, ⋯ .

We then make the following proposition:

Proposition 3: Let p* be the true probability (true prevalence rate), and let π_1 be the initial population measure on P. Let h_1 be the density function of π_1, and suppose that all of the assumptions hold. Then, a dynamic process is determined uniquely by the sequence of the density functions $\{ h_t \}_{t=1}^{+\infty}$ such that

$$h_{t+1}(\mathrm{q}) = \mathrm{F(q;\ p^*)}\cdot(1+\gamma_t)\cdot h_t(\mathrm{q}) > 0 \text{ for all } q \in \mathrm{P} \text{ and all } t = 1, 2, \cdots. \tag{12}$$

Moreover, it holds that $\lim_{t\to+\infty} \frac{h_t(\mathrm{r})}{h_t(\mathrm{p*})} = 0$ for all $r \neq p^*$.

Proof of Proposition 3: Euation (12) holds by Lemma 8, and according to Equation (12) and Proposition 2 the following holds:

$$\lim_{t\to+\infty}\frac{h_t(\mathrm{r})}{h_t(\mathrm{p}*)}=\lim_{t\to+\infty}\frac{\left[\prod_{n=1}^{t}\mathrm{F}(\mathrm{r};\,\mathrm{p}*)\cdot(1+\gamma_n)\right]^t\cdot h_1(\mathrm{r})}{\left[\prod_{n=1}^{t}\mathrm{F}(\mathrm{p}*;\,\mathrm{p}*)\cdot(1+\gamma_n)\right]^t\cdot h_1(\mathrm{p}*)}$$

$$=\lim_{t\to+\infty}\left[\frac{\mathrm{F}(\mathrm{r};\,\mathrm{p}*)}{\mathrm{F}(\mathrm{p}*;\,\mathrm{p}*)}\right]^t\cdot\frac{h_1(\mathrm{r})}{h_1(\mathrm{p}*)}=0\ \text{ for all } r\neq p*. \qquad \square$$

Even if the difference in the fitness values between the medical doctors with correct beliefs and the other medical doctors is very small, it follows from Proposition 3 that the selection will appear as a consequence of a mathematical property of the exponential function.[13] Proposition 3 holds, independent of the natural rate of the population growth rate $\gamma_t > 0$, because the rate γ_t is assumed to be common for all types of agents. Therefore, even if γ_t is variable over time, Proposition 3 still holds. For example, $\gamma_t = sin(t)$.

Example 1.(continued): When p* = 0.4, $p_\alpha = 0.3$, $p_\beta = 0.1$, set $h_1(q) = 1$ for all $q \in P$ and $\gamma_t = 0.23$ for all t. Then, we define a function H_t by $H_t(q) = \pi_t(q)/\pi_t(p^*)$ for all $q \in P$ and all t.

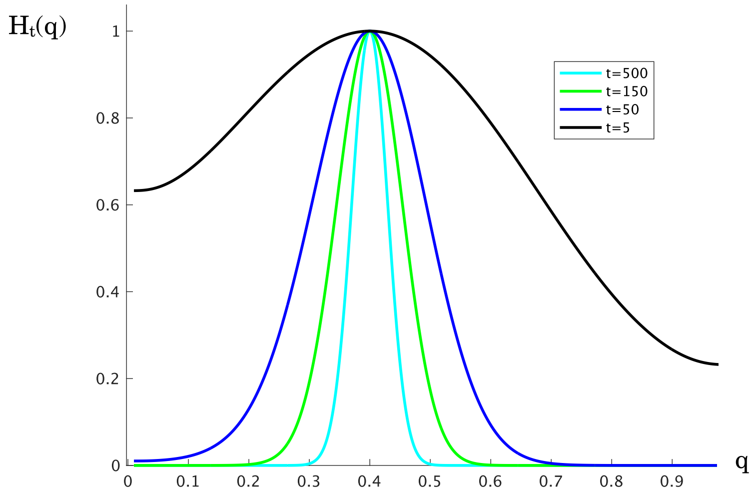

Figure 9

[13] See Nowak (2006, Ch.2, Section 2.2.1).

6. Proof of lemmas

This section proves Lemmas 1, 3, 4, 5 and 6. We need some technical claims:

Claim 1: Let g be a bounded continuous function on S in $\mathfrak{B}_X$ such that $g(x) > 0$ for all $x \in S$. For a given null set A in $\mathfrak{B}_X$, the value of the Lebesgue integral of g on S coincides with the value of the Lebesgue integral of g on S/A.

Proof of Claim 1: Claim 1 is a direct consequence of Royden (1988, Ch.4, Proposition 5(iii), Page 82). □

Claim 2: For any $Z \in \mathfrak{B}_X$ such that $\mu(Z) > 0$, there exists a compact set $Z^* \in \mathfrak{B}_X$ such that $Z^* \subset Z$ and $\mu(Z^*) > 0$.

Proof of Claim 2: Fix any $Z \in \mathfrak{B}_X$ such that $\mu(Z) > 0$. It holds by Royden (1988, Proposition 15, Ch.3, Page 63) or Ito (1984, Theorem 2.4.1, Ch.2, Page 61) that there exists a closed set $W \in \mathfrak{B}_X$ such that $W \subset Z$ and $\mu(W) > 0$. Set $V_n = [-n, n]$ for all $n = 1, 2, \cdots$. If $W \cap V_n = \emptyset$ for all n, then $\mu(W) = 0$, which is a contradiction. Hence it holds that that $W \cap V_{n^*} \neq \emptyset$ for some n^*. Thus $Z^* \equiv W \cap V_{n^*}$ is compact and $\mu(Z^*) > 0$. □

Proof of Lemma 1: Setting $X_n = (x-(1/n), x+(1/n))$ for all $n \geq 2$, it holds by the definition of $P(\theta_1 \mid x : p)$ that $P(\theta_1 \mid x : q) = \lim_{n \to +\infty} \frac{q \cdot P(X_n \mid \theta_1)}{P(X_n)}$, where $P(X_n \mid \theta_1) = \int_{X_n} f(x \mid \theta_1) d\mu$ and $P(X_n) = \int_{X_n} [q \cdot f(x \mid \theta_1) + (1-q) \cdot f(x \mid \theta_0)] d\mu = q \cdot P(X_n \mid \theta_1) + (1-q) \cdot P(X_n \mid \theta_0)$. It follows from the mean-value theorem that there exists a sequence of $\{ x_n \}$ such that: (**i**) $\lim_{n \to +\infty} x_n = x$, (**ii**) $x_n \in X_n$ for all $n \geq 2$, (**iii**) $p(X_n \mid \theta_i) = f(x_n \mid \theta_i)(2/n)$ for $i = 0, 1$ and all $n \geq 2$. Because f is continuous, we have that

$$P(\theta_1 \mid x : q) = \lim_{n \to +\infty} \frac{q \cdot P(X_n \mid \theta_1)}{P(X_n)} = \lim_{n \to +\infty} \frac{q \cdot f(x_n \mid \theta_1) \cdot (2/n)}{q \cdot f(x_n \mid \theta_1) \cdot (2/n) + (1-q) \cdot f(x_n \mid \theta_0) \cdot (2/n)}$$

$$= \frac{q \cdot f(x \mid \theta_1)}{q \cdot f(x \mid \theta_1) + (1-q) \cdot f(x \mid \theta_0)}.$$

Using almost the same manner, we can prove that $P(\theta_0 \mid x : q) = (1-q) \cdot f(x \mid \theta_0) / [q \cdot f(x \mid \theta_1) + (1-q) \cdot f(x \mid \theta_0)]$. □

Proof of Lemma 3: (**i**) Lemma 3(i) is a direct consequence of (4) and Claim 1. (**ii**) Setting $k(x) \equiv q \cdot f(x \mid \theta_1)+(1-q) \cdot f(x \mid \theta_0)$, $X_1 = \delta^{-1}(d_1)$ and $X_0 = \delta^{-1}(d_0)$, we have by (4) that

$$
\begin{aligned}
G(\delta\!: q) &= (1-q)\cdot(1-\alpha(\delta))\cdot p_\alpha + (1-q)\cdot\alpha(\delta)\cdot p_\beta + q\cdot(1-\beta(\delta))\cdot p_\alpha + q\cdot\beta(\delta)\cdot p_\beta \\
&= (1-q)\cdot p_\alpha \cdot \int_{X_0} f(x \mid \theta_0)d\mu + (1-q)\cdot p_\beta \cdot \int_{X_1} f(x \mid \theta_0)d\mu \\
&\qquad + q\cdot p_\alpha \cdot \int_{X_1} f(x \mid \theta_1)d\mu + q\cdot p_\beta \cdot \int_{X_0} f(x \mid \theta_1)d\mu \\
&= \int_{X_1} [(1-q)\cdot p_\beta \cdot f(x \mid \theta_0) + q\cdot p_\alpha \cdot f(x \mid \theta_1)]d\mu + \int_{X_0} [(1-q)\cdot p_\alpha \cdot f(x \mid \theta_0) + q\cdot p_\beta \cdot f(x \mid \theta_1)]d\mu \\
&= \int_{X_1} h(d_1 \mid x: q)\cdot k(x)d\mu + \int_{X_0} h(d_0 \mid x: q)\cdot k(x)d\mu \\
&= \int_{X_1} h(d_1 \mid x: q)d\lambda_q + \int_{X_0} h(d_0 \mid x: q)d\lambda_q = \int_X h(\delta(x) \mid x: q)d\lambda_q. \qquad \square
\end{aligned}
$$

Proof of Lemma 4: Select any $q \in P$ and let q be a subjective belief of a testing agent. (**i**) Define a function $\delta : X \to \{ d_0, d_1 \}$ by

$\delta(x) = d_1$ for all $x \in \{ x \in X : f(x \mid \theta_1)/f(x \mid \theta_0) \geq (1-q)/q \}$;

$\delta(x) = d_0$ for all $x \in \{ x \in X : f(x \mid \theta_1)/f(x \mid \theta_0) < (1-q)/q \}$.

It suffices to prove that $\{ x \in X : f(x \mid \theta_1)/f(x \mid \theta_0) < (1-q)/q\} \in \mathfrak{B}_X$, which is a direct consequence of the continuity of $L(x) \equiv f(x \mid \theta_1)/f(x \mid \theta_0)$ on X. (**ii**): Setting $k(x) \equiv q \cdot f(x \mid \theta_1)+(1-q) \cdot f(x \mid \theta_0)$, it holds that $h(d_1 \mid x: q) - h(d_0 \mid x: q) = \{ [(1-q)\cdot p_\beta \cdot f(x \mid \theta_0)+q\cdot p_\alpha \cdot f(x \mid \theta_1)] - [(1-q)\cdot p_\alpha \cdot f(x \mid \theta_0)+q\cdot p_\beta \cdot f(x \mid \theta_1)] \}/k(x) = [(p_\alpha - p_\beta)/k(x)]\cdot[q\cdot f(x \mid \theta_1) - (1-q)\cdot f(x \mid \theta_0)]$. Hence Lemma 4(ii) holds by $(p_\alpha - p_\beta)/k(x) > 0$. (**iii**) Fix any $\delta^1 \in \Delta_q$ and $\delta^2 \in \Delta$. It follows from Lemma 2(i) and Lemma 4(ii) that δ^1 has the property:

$\delta^1(x) = d_1$ for almost all x in $\{ x \in X : h(d_1 \mid x: q) > h(d_0 \mid x: q) \}$.

$\delta^1(x) = d_0$ for almost all x in $\{ x \in X : h(d_0 \mid x: q) > h(d_1 \mid x: q) \}$.

Do either for all x in $\{ x \in X : h(d_0 \mid x: q) = h(d_1 \mid x: q) \}$

This means that $h(\delta^1(x) \mid x: q) = \mathbf{max}[h(d_1 \mid x: q), h(d_0 \mid x: q)]$ for almost all $x \in X$. Because $\mathbf{max}[h(d_1 \mid x: q), h(d_0 \mid x: q)] \geq h(\delta^2(x) \mid x: q)$ for all $x \in X$, we have that

$h(\delta^1(x) \mid x: q) \geq h(\delta^2(x) \mid x: q)$ for almost all $x \in X$.

(**iv**) Lemm 4(iv) is a direct consequence of Lemma 4(iii). (**v**) Fix any $\delta^1 \in \Delta_q$, and suppose that $\delta^2 \in \Delta$ satisfies $\delta^2 \notin \Delta_q$.

Case I($\mu(C^1(q)) > 0$ and $\mu(C^0(q)) > 0$): We need a claim:

Claim 3: At least one of the following two statements holds:

(**a**) There exists $A^* \in \mathfrak{B}_X$ such that $A^* \subset C^1(q)$, $\mu(A^*) > 0$ and $\delta^2(x) = d_0$ for all $x \in A^*$.

(**b**) There exists $B^* \in \mathfrak{B}_X$ such that $B^* \subset C^0(q)$, $\mu(B^*) > 0$ and $\delta^2(x) = d_1$ for all $x \in B^*$.

Proof of Claim 3: Set $A^* = C^1(q) \cap \{ x \in X : \delta^2(x) = d_0 \}$ and $B^* = C^0(q) \cap \{ x \in X : \delta^2(x) = d_1 \}$. If $\mu(A^*) = 0$ and $\mu(B^*) = 0$, then $\mu(C^1(q) \cap \{ x \in X : \delta^2(x) = d_1 \}) = \mu(C^1(q))$ and $\mu(C^0(q) \cap \{ x \in X : \delta^2(x) = d_0 \}) = \mu(C^0(q))$. It holds by $\mu(C^1(q)) > 0$ and $\mu(C^0(q)) > 0$ that

$$\delta^2(x) = d_1 \text{ for almost all } x \in C^1(q) \text{ and } \delta^2(x) = d_0 \text{ for almost all } x \in C^0(q).$$

Hence it holds that $\delta^2 \in \Delta_q$, which is a contradiction. Thus we have that $\mu(A^*) > 0$ or $\mu(B^*) > 0$. □

First we prove Lemma 4(v) in the case (**a**) in Claim 3. It holds by Claim 2 that there exists a compact subset A in $C^1(q)$ such that $\mu(A) > 0$ and $\delta^2(x) = d_0$ for all $x \in A$. It holds by Lemma 4(ii) and Lemma 2(i) that

$$h(d_1 \mid x: q) > h(d_0 \mid x: q) \text{ for all } x \in A. \tag{13}$$

It holds by the definition of δ^1 that

$$\delta^2(x) = d_0 \text{ and } \delta^1(x) = d_1 \text{ for almost all } x \in A. \tag{14}$$

We have by (13) and (14) that $h(\delta^1(x) \mid x : q) > h(\delta^2(x) \mid x : q)$ for almost all $x \in A$. It holds by Claim 2 that there exists a compact subset A^+ in A such that $\mu(A^+) > 0$ and $h(\delta^1(x) \mid x : q) > h(\delta^2(x) \mid x : q)$ for all $x \in A^+$. Second, we can prove Lemma 4(v) in the case (**b**) by almost the same manner.

Case II ($\mu(C^1(q)) > 0$ and $\mu(C^0(q)) = 0$): We need a claim:

Claim 4: There exists $A^* \in \mathfrak{B}_X$ such that $A^* \subset C^1(q)$, $\mu(A^*) > 0$ and $\delta^2(x) = d_0$ for all $x \in A^*$.

Proof of Claim 4: Set $A^* = C^1(q) \cap \{ x \in X : \delta^2(x) = d_0 \}$. If $\mu(A^*) = 0$, then $\mu(C^1(q) \cap \{ x \in X : \delta^2(x) = d_1 \}) = \mu(C^1(q))$. It holds by $\mu(C^1(q)) > 0$ that $\delta^2(x) = d_1$ for almost all $x \in C^1(q)$. Hence it holds that $\delta^2 \in \Delta_q$, which is a contradiction. Thus we have that $\mu(A^*) > 0$. □

It holds by Claim 4 and Claim 2 that there exists a compact subset A such that $A^* \subset C^1(q)$, $\mu(A) > 0$ and $\delta^2(x) = d_0$ for all $x \in A$. It holds by Lemma 4(ii) and Lemma 2(i) that

$$h(d_1 \mid x: q) > h(d_0 \mid x: q) \text{ for all } x \in A. \tag{15}$$

It holds by the definition of δ^1 that

$$\delta^2(x) = d_0 \text{ and } \delta^1(x) = d_1 \text{ for almost all } x \in A. \tag{16}$$

We have by (15) and (16) that $h(\delta^1(x) \mid x : q) > h(\delta^2(x) \mid x : q)$ for almost all $x \in A$. It holds by Claim 2 that there exists a compact subset A^+ in A such that $\mu(A^+) > 0$ and $h(\delta^1(x) \mid x : q) > h(\delta^2(x) \mid x : q)$ for all $x \in A^+$.

Case III ($\mu(C^1(q)) = 0$ and $\mu(C^0(q)) > 0$): We can prove Lemma 4(v) in this case by almost the same manner used in the proof of the case II above.

Case IV($\mu(C^1(q)) = 0$ and $\mu(C^0(q)) = 0$): In this case, $f(x \mid \theta_1)/f(x \mid \theta_0) = (1-q)/q$ for all x. Hence we have that $\Delta_q = \Delta$, which implies that there is no $\delta^2 \in \Delta$ which satisfies $\delta^2 \notin \Delta_q$. Thus Lemma 4(v) holds in this case. (**vi**) Lemm 4(vi) is a direct consequence of Lemma 3(ii) and Lemma 4(iii, v). □

Proof of Lemma 5: Lemma 5 holds by (1), (4) and (6). □

Proof of Lemma 6: **(i)** Lemma 6(i) holds by (8) and Lemma 5. **(ii)** First, we prove that $\alpha^*(q)$ is continuous on P. Fix any $q \in P$.

Case I ($C^1(q) \neq \emptyset$): We have by (1) that $C^1(q)$ is an interval $(-\infty, c(q)]$ or $[c(q), +\infty)$, where $c(q) = L^{-1}(D(q))$. Becasue $D(q) = (1-q)/q$ and L^{-1} are continuous, $c(q) = L^{-1}(D(q))$ is continuous at q. Hence it holds by the absolute continuity of the definite integral that $\alpha^*(p)$ is continuous.

Case II ($C^1(q) = \emptyset$): If there exists an open interval Q in P such that $q \in Q$ and $C^1(r) = \emptyset$ for all $r \in Q$, then $\alpha^*(r) = 0$ for all $r \in Q$, which implies the continuity of $\alpha^*(q)$. Hence, we can assume that there exists a sequence $\{ q^n \}$ such that $\lim_n q^n = q$ and $\alpha^*(q^n) > 0$ for all n. Because $\alpha^*(q^n) > 0$ implies $C^1(q^n) \neq \emptyset$, we have by (1) that $C^1(q^n)$ is an interval $(-\infty, c(q^n)]$ or $[c(q^n), +\infty)$, where $c(q) = L^{-1}(D(q))$. If $C^1(q^n) = (-\infty, c(q^n)]$, we have by $C^1(q)$

$= \emptyset$ that $\lim c(q^n) = -\infty$ and $\lim \alpha^*(q^n) = 0 = \alpha^*(q)$. Otherwise, we have by $C^1(q) = \emptyset$ that $\lim c(q^n) = +\infty$ and $\lim \alpha^*(q^n) = 0 = \alpha^*(q)$. □

Proof of Lemma 7: Because $p \neq q$ implies $(1-p)/p \neq (1-q)/q$, Lemma 7 holds by $\mathbf{A}_8$ and the continuity of $L(x)$. □

Proof of Lemma 8: Let $\{\pi_t\}_{t=1}^{+\infty}$ be a sequence of measures satisfying (11), and let h_t be the density function satisfying (12) for $t = 1, 2, 3, \cdots$. Fix t = 1 and select any $q \in P$. Then it holds by the definition of the Radon-Nikodym density (Shilov and Gurevich, 2012, Sec.10.1, (3)) and (11) that

$$\frac{\lim_{k\to+\infty} \frac{\pi_2(B(q,k))}{\mu(B(q,k))}}{\lim_{k\to+\infty} \frac{\pi_1(B(q,k))}{\mu(B(q,k))}} = \lim_{k\to+\infty} \frac{\pi_2(B(q,k))}{\pi_1(B(q,k))} = F(q; p^*)\cdot(1+\gamma_1).$$

Because it holds by Lemma 6 and (12) that $\lim_{k\to+\infty} \frac{\pi_2(B(q,k))}{\mu(B(q,k))} = F(q; p^*)\cdot(1+\gamma_1)\cdot h_1(q)$ for all $q \in P$ and $F(q; p^*)\cdot(1+\gamma_1)\cdot h_1(q)$ is continuous and positive on P, π_2 is absolutely continuous with respect to the Lebesgue μ on P. Using an induction argument, we have that all population measures π_t ($t = 2, 3, \cdots$) are absolutely continuous with respect to the Lebesgue μ on P, and it holds that $\frac{h_{t+1}(q)}{h_t(q)} = F(q; p^*)\cdot(1+\gamma_t)$ for all $q \in P$ and all $t = 1, 2, \cdots$, where h_t be the density function of π_t for $t = 2, 3, \cdots$. □

7. References

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