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A Two-class Economy from the Multi-sectoral Perspective: The Controversy between Pasinetti and Meade–Samuelson–Modigliani Revisited

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A Two-class Economy from the Multi-sectoral Perspective: The Controversy between Pasinetti and Meade–Samuelson–Modigliani Revisited*

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Abstract

We examine the Pasinettian two-class multi-sectoral model with a microfoundation of capitalists and workers, specifically, two combinations of their behaviour. First, both act as infinitely lived agents (ILA) and second, capitalists act as ILA while workers follow overlapping generations behaviour. We analyse the switch of equilibria simultaneously with the paradox in capital theory. Pasinetti equilibrium is independent of technology and the microfoundation. Dual equilibrium depends on technology and differs by microfoundation. Numerical examples of net output/capital ratio and capital intensity imply we should analyse income and wealth distribution by the models with capital as a bundle of reproducible commodities.

JEL Classification: B51, D01, D24, E25, D33

Keywords: Pasinetti equilibrium/Dual Equilibrium, Microfoundation of Capitalists' and Workers' Behaviour, Capital Theory Paradox, Multi-sectoral Model

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1 Introduction

The distribution of income and capital (or wealth) is one of the major issues in current economic analysis. Recent research on distribution has been stimulated especially by Piketty (2014). Following him, many economists have studied the degree of concentration of the richer class's income and capital (wealth), the kinds of economic models that can fit or predict such concentration of income and capital (wealth), and other topics.

The assumptions underlying the behaviour of agents are crucial for analysis of the distribution of income and capital (wealth). Caggetti and De Nardi (2008) and De Nardi and Fella (2017) investigate the various types of general equilibrium models with multiple agents. The first type is the general equilibrium model with infinitely lived agents (ILA). The second type is overlapping generations (OLG) models, which include some elements of life-cycle structures and intergenerational links. The third type mixes features of both the ILA and OLG models. Caggetti and De Nardi (2008) and De Nardi and Fella (2017) argue that the first type cannot generate a level of wealth inequality observed in the data; the second type can explain the observed inequality much better than the first; and the third type simplifies some aspects of either model and thus, makes the analysis more tractable. In addition, they assert that the introduction of the distinction between entrepreneurs' and workers' decision-making into the models drastically improves the fit of the models to the data.

When focusing on a specific phenomenon, the simple model is very useful to understand the underlying essential features and mechanism at work. Such a simple model for the analysis of distribution of income and capital (wealth) would be a two-class model. For the two-class model, Pasinetti (1962) is an important point of reference to consider growth and distribution. This work led to intensive debates with Meade (1963, 1966), Meade and Hahn (1965), and Samuelson and Modigliani (1966a, 1966b), as part of the Cambridge capital controversy, in the 1960s.[1] Interest in the Pasinettian two-class model has been revived recently.

For example, Taylor (2014) criticises Piketty's (2014) analysis for relying on the neoclassical production function. Taylor (2014) demonstrates that euthanasia, persistence, and triumph of the rentier are all possible scenarios of the Pasinettian aggregate two-class model, whereas Piketty (2014) predicts only one scenario, triumph of the rentier. Meanwhile, Taylor's (2014) model lacks a microfoundation of agents.

Mattauch, Klenert, Stiglitz, and Edenhofer (2017) examine how public investment financed by capital tax affects the distribution of wealth when a change in substitutability between capital and labour is allowed. The basic setting of the model is based on Pasinetti (1962). In other words, the worker earns his income from wage and profits and saves for a life-cycle purpse, which implies that the worker's behaviour follows the OLG model.[2] By contrast, the capitalist earns her income solely from profit and is assumed to be an ILA. As a result, it is demonstrated that for any elasticity of substitution greater than a threshold, there exists a capital tax rate at which the capitalist disappears in the steady state, which corresponds to the dual equilibrium (DE) in the sense of Samuelson and Modigliani (1966a, 1966b). In addition, for any elasticity of substitution below the threshold there exists a cap-

[1] See also Pasinetti (1964, 1966a, 1966b, 1974). Baranzini and Mirante (2018, Chaps. 6 and 7) provide an excellent survey of the extensions of the Pasinettian two-class model.

[2] Baranzini (1991) is an early example of the models introducing the OLG into the Pasinettian two-class model.

ital tax rate at which the worker disappears in the steady state, which corresponds to the anti-dual equilibrium (ADE) in the sense of Darity (1981). Furthermore, both the capitalist and worker co-exist in the steady state below these capital tax rates, which corresponds to the Pasinetti equilibrium (PE), shown in Pasinetti (1962) and named in Samuelson and Modigliani (1966a, 1966b).[3]

Zamparelli (2016) analyses the Pasinettian two-class economy in the neo-classical framework. He assumes a production function with constant elasticity of substitution (CES) between capital and labour and no microfoundation of capitalist's and worker's behaviour. The capitalist earns her income solely from profit and the worker from wage and profit. As a result, if the capitalist's saving rate is higher than the worker's and the elasticity of substitution is high enough to ensure endogenous growth, ADE exists. In addition, he shows that capital tax can favourably improve the distribution to the worker in the steady state.

Stiglitz (2016) points out *new* stylised facts on growth and accumulation, and asserts that the standard neo-classical models cannot explain the recent movement of the ratio of wealth to income, even taking technical change into consideration.[4] Then, Stiglitz (2016) assumes the Pasinettian two-class economy in which the capitalist is the ILA and the worker's behaviour follows the OLG model. The feature of his two-class model is to introduce land as a factor of production in the model, as taking land rent and exploitation rent into account better explains the recent movement of the ratio of wealth to income.

Sasaki (2018) investigates the existence and stability of the steady states obtained by the Pasinettian two-class economy in which the capitalist is an ILA, the worker's behaviour follows the OLG model, and there is a CES production function. It is shown that although PE and DE exist depending on the combinations of parameters, PE is stable under reasonable values of the parameters. Furthermore, Taylor, Foley, and Rezai (2019) consider a Pasinettian two-class economy in which growth is demand driven and no microfoundation of agents is formulated. Taylor et al. (2019) examine the existence and stability of the PE and DE obtained as the steady states in the demand-driven growth model. In a context of the richest 1% of US households receiving about 7% of wages, Taylor et al. (2019) interestingly consider a case in which capitalists, who have a higher rate of saving than that of workers, receive some wage income besides profit. Then, it is shown that the DE obtained in this case is unstable.

To consider social security, Michl and Foley (2004), Michl (2007, 2009) construct a Pasinettian two-class growth model in which the capitalist is an ILA and the worker's be-

[3]Mattauch, Edenhofer, Klenert, and Bénard (2016) investigate the distribution of wealth in relation to public investment in a model with high-income households that are ILA and middle-income households that follow the OLG model. Their analysis focuses on the equilibrium at which both types of households survive in the steady state (i.e. PE).

[4]In contrast to Kaldor's (1961) old stylised facts, the new ones proposed by Stiglitz (2016) are:

- there is growing inequality in both wages and capital income (wealth), and growing inequality overall;
- wealth is more unequally distributed than wages;
- average wages have stagnated, even as productivity has increased, so the share of capital has increased;
- the wealth ratio has increased significantly;
- the return on capital has not declined, even as the wealth–income ratio has increased.

haviour follows the OLG model. They show the existence of PE and DE in the model, and analyse various effects of social security. For example, Michl and Foley (2004) show that an unfunded social security system relying on payroll taxes reduces workers' lifetime welfare and saving, since the change in their saving affects the level of the share of capital owned by the workers but not the rate of economic growth in PE. The change in worker's savings has no effect on the path of capital owned by capitalists in the model. Therefore, the decrease in the level of workers' share of capital reduces the overall level of capital, output, and employment without affecting the rate of economic growth. This is called the level effect. The effect is mitigated by the presence of a reserve fund. Their model identifies the social security reserve fund as a potential vehicle for generating capital accumulation and effecting progressive redistribution of wealth.

The above-mentioned models with a microfoundation of agents assume the neo-classical 'well-behaved' production function with capital as the primary factor of production or are *de-facto* one-commodity models. As clarified in the Cambridge capital controversy, the assumption of the neo-classical well-behaved production function with capital as the primary factor of production by hypothesis excludes some phenomena which may arise if capital is regarded as a bundle of reproducible and heterogeneous commodities.[5] Furthermore, the controversy reveals that the results obtained by the one-commodity model do not necessarily hold in a model with multiple commodities (see, e.g., Harcourt, 1972).

As highlighted by Piketty (2014), capital in modern capitalist economies typically consists of reproducible and heterogeneous commodities. It would be significant to analyse growth and distribution in the Pasinettian two-class economy under the assumption of multiple commodities and capital as a bundle of reproducible and heterogeneous commodities. In this study, we examine the Pasinettian two-class economy by a sort of activity analysis (i.e. Leontief–Sraffa model). Since, as already mentioned, the kinds of assumptions made about the micro-behaviour of agents are crucial for our purpose, we consider two combinations of microfoundation of capitalists' and workers' behaviour: first, both capitalists and workers are ILA; and second, capitalists are ILA and workers' behaviour follows the OLG model. The scenarios cover all combinations used in the abovementioned models. However, our examination is confined to the steady states obtained in PE and DE.

The first characteristic of our study is that we analyse the switch of the type of equilibria (from PE to DE, or vice versa) simultaneously with paradoxical phenomena in capital theory (reswitching of technique and reverse capital deepening). This analysis is certainly impossible in models which assume the neo-classical well-behaved production function with capital as the primary factor of production. This is because, as mentioned, the possibility of such paradoxical phenomena arising is *by hypothesis* excluded from the models. Although Pasinetti (1962) and Samuelson and Modigliani (1966a, 1966b) use aggregate macroeconomic models, Morishima (1969) constructs a multi-sectoral general equilibrium model to analyse the properties of the steady-state path of PE and DE. Subsequently, Hosoda (1989) extends Morishima's model to analyse the switch of the type of equilibria simultaneously with the capital theory paradoxes. Since capitalists' and workers' saving rates are assumed to be exogenously given in both Morishima's (1969) and Hosoda's (1989) models, we further develop

[5] Stiglitz (2016) mentions the possibility of such phenomena arising, yet assumes a perfectly neo-classical technology in his model.

them to consider the abovementioned combinations of the microfoundation of capitalists' and workers' behaviour.

In analysing the distribution of income/capital (wealth), the movements of the ratio of output to capital and the profit (or the wage) share are important variables to be determined in a model. Piketty (2015) argues that:

> the right model to think about rising capital–income ratios and capital shares in recent decades is a multi-sector model of capital accumulation, with substantial movements in relative prices, and with important variation in bargaining power.

His claim is based on the reflection, which Stiglitz (2016) also asserts, that the recent movements of the ratio of capital to output and the profit share cannot be explained by standard neo-classical models in which a well-behaved production function with capital as the primary factor is assumed. Although we do not consider the variation of bargaining power, our multi-sectoral model based on Morishima (1969) and Hosoda (1989) shown in Section 3 sufficiently satisfies Piketty's claim. This is because the movements of the ratio of net output to capital and the income share that are inconsistent with the principle of marginal productivity of capital are obtained by the Leontief–Sraffa model in which capital is composed of a bundle of reproducible and heterogeneous commodities. This is the second characteristic of our model.

The rest of this paper is organised as follows. Section 2 briefly reviews the concept of the long-run competitive equilibrium in Hosoda (1989), on which our model is based. Hosoda's (1989) model has one degree of freedom, as in Sraffa (1960), and thus, the rate of profit is treated as exogenously given throughout his study. We follow his treatment of the rate of profit in our model. Therefore, the rate of economic growth in the steady states is obtained as the function of the rate of profit, unlike in Pasinetti (1962, 1974) and Samuelson and Modigliani (1966a, 1966b), whose rate of profit is the endogenous variable and rate of economic growth is the exogenous variable (*natural* rate of economic growth). This is a device to analyse the paradoxes in capital theory, together with economic growth. Section 3 presents our multi-sectoral two-class model with a microfoundation of capitalists and workers. We analyse the relationship between the rates of economic growth and profit obtained in each combination of the microfoundation of capitalist and worker. We show the existence of PE and DE and analyse the properties of the steady states. Section 4 presents numerical examples to examine the working of the model proposed in Section 3; we analyse the movements of the ratio of net output to capital and the capital intensity in relation to the change in the rate of profit. Section 5 presents concluding remarks.

2 Long-run Competitive Equilibrium

The main purpose of this section is to review Pasinetti's (1962) two-class growth model from a multi-sectoral perspective. We focus on the basic structure of the models in Morishima (1969) and Hosoda (1989).

We assume that the primary factor of production is labour alone, joint production is absent, and the number of reproducible commodities is n; the jth sector has $m(j) \geqq 1$ processes. Column vectors $\boldsymbol{A}_{j1}, \cdots, \boldsymbol{A}_{jm(j)} \in \mathbb{R}^n_+$ for $j = 1, \cdots, n$ denote processes of the jth sector to produce a unit of the commodity. We define an *input matrix* as follows:

$$\boldsymbol{A} \equiv \left(\boldsymbol{A}_{11}, \cdots, \boldsymbol{A}_{1m(1)}, \boldsymbol{A}_{21}, \cdots, \boldsymbol{A}_{2m(2)}, \cdots, \boldsymbol{A}_{n1}, \cdots, \boldsymbol{A}_{nm(n)}\right) \in \mathbb{R}_{+}^{n \times m},$$

where $m = \sum_{j=1}^{n} m(j)$. Letting $\boldsymbol{L}_{jk}$ be the labour input coefficient corresponding to the process $\boldsymbol{A}_{jk}$ for $k \in \{1, \cdots, m(j)\}$, we define such a *labour input vector* as follows:

$$\boldsymbol{L} \equiv \left(L_{11}, \cdots, L_{1m(1)}, L_{21}, \cdots, L_{2m(2)}, \cdots, L_{n1}, \cdots, L_{nm(n)}\right) \in \mathbb{R}_{+}^{m}.$$

Moreover, we define $\boldsymbol{E}$ as

$$\boldsymbol{E} \equiv (\underbrace{\boldsymbol{e}_1, \cdots, \boldsymbol{e}_1,}_{m(1)} \underbrace{\boldsymbol{e}_2, \cdots, \boldsymbol{e}_2,}_{m(2)} \cdots, \underbrace{\boldsymbol{e}_n, \cdots, \boldsymbol{e}_n}_{m(n)}) \in \mathbb{R}^{n \times m}$$

where $\boldsymbol{e}_j \in \mathbb{R}^n$ denote the vector, the jth element of which is unity and the others zero.

We choose only one process from each sector. Define a column vector $\boldsymbol{x} \equiv \left(x_{1k(1)}, x_{2k(2)} \cdots, x_{nk(n)}\right) \in \mathbb{R}_{+}^{n}$, indicating that the jth sector utilises only one process $k(j)$. Furthermore, we define the following vector in accordance with $\boldsymbol{A}$:

$$\boldsymbol{q} \equiv (\underbrace{0, \cdots, x_{1k(1)} \cdots, 0,}_{m(1)} 0, \cdots, 0, \underbrace{0 \cdots, x_{nk(n)}, \cdots, 0}_{m(n)}) \in \mathbb{R}_{+}^{m}.$$

In addition, we define such a matrix as

$$\boldsymbol{A}_{\iota} \equiv \left(\boldsymbol{A}_{1k(1)}, \boldsymbol{A}_{2k(2)}, \cdots, \boldsymbol{A}_{nk(n)}\right) \in \mathbb{R}_{+}^{n \times n},\ k(j) \in \{1, \cdots, m(j)\}, \text{ for } j = 1, \cdots, n.$$

The matrix is termed a *technical coefficient matrix*. Corresponding to $\boldsymbol{A}_{\iota}$, we construct such a labour input vector as $\boldsymbol{L}_{\iota} \equiv \left(L_{1k(1)}, L_{2k(2)}, \cdots, L_{nk(m)}\right) \in \mathbb{R}_{+}^{n}$, which is termed a *technical labour input vector*. There are $\prod_{j=1}^{n} m(j)$ such matrices and vectors. We make pairs of the technical coefficient matrix and the technical labour input vector $(\boldsymbol{A}_{\iota}, \boldsymbol{L}_{\iota})$, and index them, such as $\alpha, \beta, \cdots$. Define the set $\Delta \equiv \{\alpha, \beta, \cdots\}$. Of course, $|\Delta| = \prod_{j=1}^{n} m(j)$ holds, where $|\cdot|$ is the number of elements of the set.

Here, we make the following assumptions.

Assumption 1 (A1): Every commodity is basic for any technique in the sense of Sraffa (1960).

Assumption 2 (A2): For every $\boldsymbol{u} \in \mathbb{R}_{+}^{n}$, there exists $\overline{g} > 0$ and $\boldsymbol{q} \in \mathbb{R}_{+}^{m}$ such that

$$\left[\boldsymbol{E} - (1 + \overline{g})\, \boldsymbol{A}\right] \boldsymbol{q} = \boldsymbol{u}.$$

Assumption 3 (A3): The period of production of all goods is one unit of time and labour is an indispensable factor of production.

A1 implies that every commodity is required as an input directly or indirectly to produce a unit of any commodity, which means that every technical matrix is indecomposable. **A2** implies the existence of $\bar{g} \in (0, G)$ and $\boldsymbol{q} \geq \mathbf{0}$, such that $[\boldsymbol{E} - (1 + \bar{g}) \boldsymbol{A}] \boldsymbol{q} > \mathbf{0}$ ($G \equiv \frac{1}{\lambda^*} - 1$ and λ^* is the minimum Frobenius root of technical coefficient matrices). Certainly, G equals the maximum rate of profit R attainable in the given set Δ. In other words, **A2** implies that $\boldsymbol{A}$ is *productive* in the sense that positive net output can be produced. **A1**–**A3** are the mild assumptions usually made in the Leontief–Sraffa production models.

We assume perfectly competitive markets, so that the rate of profit r_t and the wage rate w_t established in time t are uniform. Matrix $\boldsymbol{A}$ represents the technology available at each time $t = 1, 2, \cdots$.[6] The activity level at time t is denoted by $\boldsymbol{q}_t$ (or $\boldsymbol{x}_t$). The amount of input commodities necessary to produce $\boldsymbol{q}_t$ (or $\boldsymbol{x}_t$) is denoted by the vector of $\boldsymbol{A}\boldsymbol{q}_t$ (or $\boldsymbol{A}_\iota \boldsymbol{x}_t$), and those input commodities necessary are produced at time $t-1$, which can be purchased at prices $\boldsymbol{p}_{t-1} \in \mathbb{R}^n_+$ at the beginning of time t. The amount of labour necessary to produce $\boldsymbol{q}_t$ (or $\boldsymbol{x}_t$) is denoted by $\boldsymbol{L}\boldsymbol{q}_t$ (or $\boldsymbol{L}_\iota \boldsymbol{x}_t$). The uniform wage rate w_t is *ex-post* paid to the workers.

Suppose that our economy is composed of agents belonging to the capitalist class and those belonging to the worker class. Since we assume that all members belonging to each class have the same preferences, the behaviour of each class as a whole can be represented by that of any of its constituents. Therefore, we pay attention to a single capitalist and a single worker.

Pasinetti (1962) argues that the worker owns a part of the stock of capital when he saves and thus, receives a part of profits as well as wages. Therefore, Pasinetti, in correcting Kaldor's (1956) saving functions asserts that the saving function of the capitalist should be formulated as $S^c_t = s_c P^c_t$ and that of the worker as $S^w_t = s_w (W_t + P^w_t)$, where $S^c_t, S^w_t, P^c_t, P^w_t, W_t, s_c,$, and s_w denote the capitalist's saving, the worker's saving, the amount of profits distributed to the capitalist, that distributed to the worker, the total amount of wages, the constant saving rate of the capitalist, and that of the worker, respectively. In other words, the total saving function is given by $S_t \equiv S^c_t + S^w_t = s_c P^c_t + s_w (W_t + P^w_t)$. In addition, $0 \leqq s_w < s_c \leqq 1$ are assumed, and $P^c_t + P^w_t = P_t$ holds, where P_t denotes the total profits generated in the economy. Since P^c_t is the capitalist's total income and $W_t + P^w_t$ is the worker's total income, it is obvious that the capitalist's consumption demand and the worker's consumption demand for each good are proportional to their incomes, and the proportionality is given by constant $1 - s_c$ in the capitalist's consumption demand function and constant $1 - s_w$ in the worker's consumption demand function.

The problem to be addressed here is how to divide P_t into P^c_t and P^w_t. Pasinetti (1962) proposes the conditions for the division of profits. The conditions are indicated as follows.

Condition 1 (C1): The stock of capital owned by the capitalist and that by the worker grow at the same rate:

$$\frac{\dot{K}_t}{K_t} = \frac{\dot{K}^c_t}{K^c_t} = \frac{\dot{K}^w_t}{K^w_t} \text{ for } \forall t,$$

[6] Here, we do not consider technical progress in that the row or the column of $\boldsymbol{A}$ and $\boldsymbol{L}$ changes as time goes by. In other words, m and n are fixed over time.

where K_t^c, K_t^w denote the stock of capital owned by the capitalist and that by the worker, respectively, at time t.

Condition 2 (C2): The rate of profit irrespective of its owner is uniform:

$$\frac{P_t}{K_t} = \frac{P_t^c}{K_t^c} = \frac{P_t^w}{K_t^w} \text{ for } \forall t,$$

By combining **C1** with **C2** and using the definitional relationship (i.e. $\dot{K}_t^c \equiv S_t^c$ and $\dot{K}_t^w \equiv S_t^w$), we obtain Condition 3.

Condition 3 (C3): The following relationship is satisfied:

$$\frac{P_t^w}{S_t^w} = \frac{P_t^c}{S_t^c} \text{ for } \forall t.$$

Following **C3** and the above defined saving functions of the capitalist and worker, we obtain

$$P_t^w = \frac{s_w}{s_c - s_w} W_t = \frac{s_w}{s_c - s_w} w_t \boldsymbol{L} \boldsymbol{q}_t, \tag{1}$$

$$P_t^c \equiv P_t - P_t^w = r_t \boldsymbol{p}_{t-1} \boldsymbol{A} \boldsymbol{q}_t - \frac{s_w}{s_c - s_w} w_t \boldsymbol{L} \boldsymbol{q}_t. \tag{2}$$

As is well known from (2), we can obtain the PE or DE, depending on whether $r_t \boldsymbol{p}_{t-1} \boldsymbol{A} \boldsymbol{q}_t \lesseqqgtr \frac{s_w}{s_c - s_w} w_t \boldsymbol{L} \boldsymbol{q}_t$ hold, which we closely analyse in Section 3. The capitalist and worker co-exist in PE, while in DE, the capitalist disappears and the entire stock of capital is owned by the worker.

Based on (2), Hosoda (1989) assumes that the capitalist's consumption demand $\boldsymbol{c}_t \in \mathbb{R}_+^n$ is given as follows:

$$\boldsymbol{c}_t = \max \left[r_t \boldsymbol{p}_{t-1} \boldsymbol{A} \boldsymbol{q}_t - \frac{s_w}{s_c - s_w} w_t \boldsymbol{L} \boldsymbol{q}_t, 0 \right] \frac{1 - s_c}{\boldsymbol{p}_t \boldsymbol{\phi}(\boldsymbol{p}_t)} \boldsymbol{\phi}(\boldsymbol{p}_t), \tag{3}$$

where $\boldsymbol{\phi}(\boldsymbol{p}_t) \in \mathbb{R}_+^n$ denotes the capitalist's consumption basket per unit of her income, which is assumed to be the function of prices. Based on (1), the worker's consumption demand $\boldsymbol{d}_t \in \mathbb{R}_+^n$ is given as follows:

$$\boldsymbol{d}_t = \min \left[\frac{s_w}{s_c - s_w} w_t \boldsymbol{L} \boldsymbol{q}_t, r_t \boldsymbol{p}_{t-1} \boldsymbol{A} \boldsymbol{q}_t \right] \frac{1 - s_w}{\boldsymbol{p}_t \boldsymbol{\varphi}(\boldsymbol{p}_t)} \boldsymbol{\varphi}(\boldsymbol{p}_t), \tag{4}$$

where $\boldsymbol{\varphi}(\boldsymbol{p}_t) \in \mathbb{R}_+^n$ denotes the worker's consumption basket per unit of his income, which is, as in the capitalist consumption demand function, assumed to be the function of prices. It is assumed that both $\boldsymbol{\phi}(\boldsymbol{p}_t)$ and $\boldsymbol{\varphi}(\boldsymbol{p}_t)$ are homogeneous functions of degree zero with respect to prices, and $\boldsymbol{p}_t \boldsymbol{\phi}(\boldsymbol{p}_t) \neq 0$ and $\boldsymbol{p}_t \boldsymbol{\varphi}(\boldsymbol{p}_t) \neq 0$ for $\forall t$. The capitalist's consumption demand and the worker's consumption demand are homothetic functions.

Based on these assumptions, we construct the two-class general equilibrium model as follows:

$$
\begin{cases}
\boldsymbol{p}_t\boldsymbol{E} \leqq (1+r_t)\,\boldsymbol{p}_{t-1}\boldsymbol{A} + w_t\boldsymbol{L}, \\
\boldsymbol{p}_t\boldsymbol{E}\,\boldsymbol{q}_{t-1} = (1+r_t)\boldsymbol{p}_{t-1}\boldsymbol{A}\,\boldsymbol{q}_{t-1} + w_t\boldsymbol{L}\,\boldsymbol{q}_{t-1}, \\
\boldsymbol{E}\,\boldsymbol{q}_{t-1} \geqq \boldsymbol{A}\,\boldsymbol{q}_t + \boldsymbol{c}_t + \boldsymbol{d}_t, \\
\boldsymbol{p}_t\boldsymbol{E}\,\boldsymbol{q}_{t-1} = \boldsymbol{p}_t\boldsymbol{A}\,\boldsymbol{q}_t + \boldsymbol{p}_t\boldsymbol{c}_t + \boldsymbol{p}_t\boldsymbol{d}_t, \\
\boldsymbol{p}_t\boldsymbol{E}\,\boldsymbol{q}_{t-1} > 0, \\
\boldsymbol{p}_t, \boldsymbol{q}_t \geq \boldsymbol{0}, w_t \geq 0, \text{ for all } t.
\end{cases} \tag{5}
$$

Assuming that all sectors grows at an exogenously given rate of $g \in [0, G)$, the economy is on a balanced growth path such that $\boldsymbol{p}_{t-1} = \boldsymbol{p}_t, \boldsymbol{q}_t = (1+g)\,\boldsymbol{q}_{t-1}, \boldsymbol{c}_t = \boldsymbol{c}_{t-1}, \boldsymbol{d}_t = \boldsymbol{d}_{t-1}, w_{t-1} = w_t$, and $r_{t-1} = r_t$ hold for $\forall t$. As mentioned in Section 1, we treat the rate of profit as the exogenous variable throughout this study. Then, the cost-minimising technique and thus, the equilibrium prices are determined independently of the capitalist's consumption demand and the worker's consumption demand, owing to the non-substitution theorem. Let $\boldsymbol{A}^* \in \mathbb{R}_+^{n\times n}$ and $\boldsymbol{L}^* \in \mathbb{R}_+^n$ be the cost-minimising technical coefficient matrix and the cost-minimising technical labour input vector, which are termed the *long-run competitive equilibrium technical coefficient matrix* and the *long-run competitive equilibrium labour input vector*, respectively, for the given rate of profit $r \in [0, R)$. In other words, the given rate of profit determines the cost-minimising technique $(\boldsymbol{A}_\iota, \boldsymbol{L}_\iota)$, independently of the size and structure of the final demand, for $\iota \in \Delta$. Given the rate of profit, the long-run competitive equilibrium technical coefficient matrix and the long-run competitive equilibrium labour input vector are represented by $\boldsymbol{A}^* = \boldsymbol{A}_\iota$ and $\boldsymbol{L}^* = \boldsymbol{L}_\iota$, respectively, for $\iota \in \Delta$.

Exogenising $r \in [0, R)$, we can normalise the prices by $w = 1$. Then, the price $\boldsymbol{p}^* \in \mathbb{R}_+^n$, termed the *long-run competitive equilibrium price vector*, is determined. The non-negativeness of the price is guaranteed by **A2**. The activity vector $\boldsymbol{x} \in \mathbb{R}_+^n$ satisfying (6), which is based on (3), (4), and (5), is termed the *long-run competitive equilibrium activity vector*:

$$
\begin{cases}
\boldsymbol{p}^* \leqq (1+r)\,\boldsymbol{p}^*\boldsymbol{A}^* + \boldsymbol{L}^*, \\
\boldsymbol{p}^*\boldsymbol{x} = (1+r)\boldsymbol{p}^*\boldsymbol{A}^*\boldsymbol{x} + \boldsymbol{L}^*\boldsymbol{x}, \\
\boldsymbol{x} \geqq (1+g)\,\boldsymbol{A}^*\boldsymbol{x} + \boldsymbol{c} + \boldsymbol{d}, \\
\boldsymbol{p}^*\boldsymbol{x} = (1+g)\,\boldsymbol{p}^*\boldsymbol{A}^*\boldsymbol{x} + \boldsymbol{p}^*\boldsymbol{c} + \boldsymbol{p}^*\boldsymbol{d}, \\
\boldsymbol{p}^*\boldsymbol{x} > 0 \\
\boldsymbol{x} \geq \boldsymbol{0},
\end{cases} \tag{6}
$$

where

$$\begin{cases} \boldsymbol{c} = \max\left[r\boldsymbol{p}^*\boldsymbol{A}^*\boldsymbol{x} - \frac{s_w}{s_c-s_w}\boldsymbol{L}^*\boldsymbol{x}, \boldsymbol{0}\right]\frac{1-s_c}{\boldsymbol{p}^*\boldsymbol{\phi}(\boldsymbol{p}^*)}\boldsymbol{\phi}\left(\boldsymbol{p}^*\right), \\ \boldsymbol{d} = \min\left[\boldsymbol{L}^*\boldsymbol{x} + \frac{s_w}{s_c-s_w}\boldsymbol{L}^*\boldsymbol{x}, r\boldsymbol{p}^*\boldsymbol{A}^*\boldsymbol{x}\right]\frac{1-s_w}{\boldsymbol{p}^*\boldsymbol{\varphi}(\boldsymbol{p}^*)}\boldsymbol{\varphi}\left(\boldsymbol{p}^*\right). \end{cases} \tag{7}$$

Hosoda (1989) proves the existence of $\boldsymbol{x}^*$, together with $\boldsymbol{p}^*$, in the set $K \equiv \{\boldsymbol{x} \geqq \boldsymbol{0} | \boldsymbol{L}^*\boldsymbol{x} = 1\}$ by applying Brouwer's fixed point theorem. Therefore, the rate of economic growth can be obtained as the function of the rate of profit as follows:

$$g\left(r\right) = \begin{cases} s_c r \text{ if } r\boldsymbol{p}^*\boldsymbol{A}^*\boldsymbol{x}^* > \frac{s_w}{s_c-s_w} \text{ (the PE)}, \\ s_w\left(r + \frac{1}{\boldsymbol{p}^*\boldsymbol{A}^*\boldsymbol{x}^*}\right) \text{ if } r\boldsymbol{p}^*\boldsymbol{A}^*\boldsymbol{x}^* \leqq \frac{s_w}{s_c-s_w} \text{ (the DE)}. \end{cases} \tag{8}$$

The first function of (8) is substantially equivalent to the *Cambridge equation* or the *Pasinetti theorem* (Samuelson and Modigliani, 1966a, 1966b; Pasinetti, 1974). Note that the *Cambridge equation* obtains the rate of profit given the rate of economic growth (*natural* rate of economic growth) in the two-class economy, whereas the first function of (8) determines the rate of economic growth given the rate of profit. However, it has the same macroeconomic implication as the *Cambridge equation*: the worker's saving rate and technology have no effect on the relationship between the rates of economic growth and profit in the PE, and the relationship is determined solely by the capitalist's saving rate. Similarly, the second function of (8) is substantially equivalent to the DE in Samuelson and Modigliani (1966a, 1966b) and has the same macroeconomic implication: the worker's saving rate and the technology affect the relationship. We should note that the form of $g\left(r\right)$ in DE depends on the choice of numéraire except when the so-called organic composition of capital is uniform among all sectors whereas the form in PE is not.

When PE holds, ($r\boldsymbol{p}^*\boldsymbol{A}^*\boldsymbol{x}^* > \frac{s_w}{s_c-s_w}$), we have:

$$s_c r > s_w\left(r + \frac{1}{\boldsymbol{p}^*\boldsymbol{A}^*\boldsymbol{x}^*}\right).$$

Similarly, when DE holds (i.e. $r\boldsymbol{p}^*\boldsymbol{A}^*\boldsymbol{x}^* \leqq \frac{s_w}{s_c-s_w}$), we have

$$s_c r \leqq s_w\left(r + \frac{1}{\boldsymbol{p}^*\boldsymbol{A}^*\boldsymbol{x}^*}\right).$$

As Hosoda (1989) shows, either $r\boldsymbol{p}^*\boldsymbol{A}^*\boldsymbol{x}^* > \frac{s_w}{s_c-s_w}$ or $r\boldsymbol{p}^*\boldsymbol{A}^*\boldsymbol{x}^* \leqq \frac{s_w}{s_c-s_w}$ holds but not both for $\boldsymbol{x}^* \in K$. We can obtain a higher rate of economic growth for $\forall r \in [0, R)$ in the equilibrium.

3 Multi-sectoral Two-class Models with Microfoundation of Agents

In this section, we provide the microfoundation of the behaviour of capitalist and worker with the two-class model, so that we can endogenise the allocation of their incomes between consumption and saving. As mentioned in Section 1, we consider two combinations of their behaviour; first, both the capitalist and worker act as ILA, and second, the capitalist acts as an ILA while the worker's behaviour follows that of the OLG model. In other words, two combinations of microfoundation are represented by (capitalist, worker) = (ILA, ILA), (ILA, OLG).

Hereafter, we assume that consumption good consists of a single commodity (commodity 1), although there are n kinds of reproducible and heterogeneous commodities. This is a simplification often made in models in which multiple commodities exist to avoid unnecessary complications from the existence of multiple consumption goods (see, e.g. Harcourt, 1972). By so doing, we can concentrate on the analysis of the problems related to the existence of capital that consists of a bundle of reproducible and heterogeneous commodities. Following the two-class models with the microfoundation referred to in Section 1, we use the log utility function. Moreover, the assumptions made in Section 2 with respect to the technology and the time structure of the model are kept unchanged.

3.1 Optimal behaviour in the case of (capitalist, worker) = (ILA, ILA)

In this subsection, we consider the case in which the combination of the microfoundation of the capitalist and worker is represented by (capitalist, worker) = (ILA, ILA) and we analyse the properties of PE and DE.

3.1.1 Optimal behaviour of the capitalist in (ILA, ILA)

Since the capitalist is assumed to act as an ILA, she maximises her utility subject to the intertemporal budget constraint. The capitalist solves the following problem:

$$\max_{c^c_{t,1}, \boldsymbol{q}_{t+1}} U^c \equiv (1-\rho_c) \sum_{t=1}^{\infty} \rho_c^{t-1} \ln c_{t,1}$$

$$\text{s.t. } \boldsymbol{p}_t \boldsymbol{E} \left((1-\theta_{t+1})\, \boldsymbol{q}_{t+1} \right) + \boldsymbol{p}_t \boldsymbol{c}_t \leqq r_t \boldsymbol{p}_{t-1} \boldsymbol{A} \left((1-\theta_t)\, \boldsymbol{q}_t \right),$$

$$\text{given } \{r_t\}_{t=1,2,\cdots}, \ \{\theta_t\}_{t=1,2,\cdots}, \ \boldsymbol{p}_0 \geq \boldsymbol{0},$$

where $U^c, \rho_c \in (0,1)$, and $\theta_t \in (0,1]$ denote the capitalist's utility function, the discount rate of the capitalist, and the worker's claim to a share of the profit, respectively, at time t. Furthermore, $\boldsymbol{c}_t \equiv (c_{t,1}, 0, \cdots, 0) \in \mathbb{R}^n_+$ is the capitalist's consumption vector.

We assume that the transversality condition is satisfied (i.e. $\boldsymbol{p}_T \boldsymbol{E} \boldsymbol{q}_{T+1} = 0$ as $T \to \infty$). We can identify the long-run competitive equilibrium technique $\boldsymbol{A}^*$ for the given rate of profit $r_t \in [0, R)$, based on the non-substitution theorem. Therefore, the solution of the problem is

$$\boldsymbol{p}^*_t \boldsymbol{c}_t = (1-\rho_c)\, r_t \boldsymbol{p}^*_{t-1} \boldsymbol{A}^* \left((1-\theta_t)\, \boldsymbol{x}_t \right), \text{ i.e.} \tag{9}$$

$$c_{t,1} = \frac{(1-\rho_c)\, r_t \boldsymbol{p}^*_{t-1} \boldsymbol{A}^* \left((1-\theta_t)\, \boldsymbol{x}_t \right)}{p^*_{t,1}},$$

where $p^*_{t,1}$ denotes the price of commodity 1 determined by r_t. (9) implies that the capitalist consumes commodity 1 proportionally to her income $r_t \boldsymbol{p}^*_{t-1}\boldsymbol{A}^*\left((1-\theta_t)\,\boldsymbol{x}_t\right)$. Therefore, the capitalist's saving rate is ρ_c.

3.1.2 Optimal behaviour of the worker in (ILA, ILA)

The worker, like the capitalist, maximises his utility subject to the intertemporal budget constraint. Then, he solves the following problem:

$$\max_{c^w_{t,1},\boldsymbol{q}_{t+1}} U^w \equiv (1-\rho_w)\sum_{t=1}^{\infty}\rho_w^{t-1}\ln d_{t,1}$$

$$\text{s.t. } \boldsymbol{p}_t\boldsymbol{E}\left(\theta_{t+1}\boldsymbol{q}_{t+1}\right)+\boldsymbol{p}_t\boldsymbol{d}_t \leqq \boldsymbol{L}\boldsymbol{q}_t + r_t\boldsymbol{p}_{t-1}\boldsymbol{A}\left(\theta_t\boldsymbol{q}_t\right),$$

$$\text{given } \{r_t\}_{t=1,2,\cdots}, \{\theta_t\}_{t=1,2,\cdots},\ \boldsymbol{p}_0 \geq \boldsymbol{0},$$

where U^w and $\rho_w \in (0,1)$ denote the worker's utility function and the discount rate of the worker, respectively.[7] Similarly, in the case of the capitalist, $\boldsymbol{d}_t \equiv (d_{t,1},0,\cdots,0)\in\mathbb{R}^n_+$ is the worker's consumption vector. Note that the wage rate is normalised as $w_t=1$. In addition, the following assumption is made:

$$\rho_w < \rho_c. \tag{10}$$

(10) means that the capitalist's saving rate must be higher than the worker's. As is mentioned in Section 2, it is nothing but the assumption made by Pasinetti (1962).

If the transversality condition is satisfied, as in the case of the capitalist's problem, the solution of the problem is

$$\boldsymbol{p}^*_t\boldsymbol{d}_t = (1-\rho_w)\left(\boldsymbol{L}^*\boldsymbol{x}_t + r_t\boldsymbol{p}^*_{t-1}\boldsymbol{A}^*\left(\theta_t\boldsymbol{x}_t\right)\right), \text{ i.e.} \tag{11}$$

$$d_{t,1} = \frac{(1-\rho_w)\left(\boldsymbol{L}^*\boldsymbol{x}_t + r_t\boldsymbol{p}^*_{t-1}\boldsymbol{A}^*\left(\theta_t\boldsymbol{x}_t\right)\right)}{p^*_{t,1}}.$$

(11) demonstrates that the worker consumes commodity 1 proportionally to his income $\boldsymbol{L}^*\boldsymbol{x}_t + r_t\boldsymbol{p}^*_{t-1}\boldsymbol{A}^*\left(\theta_t\boldsymbol{x}_t\right)$, and the worker's saving rate is ρ_w.

[7] Although it is assumed that both the capitalist and worker control $\boldsymbol{q}_{t+1}$, we do not need to apply the differential game to our problem. This is because the non-substitution theorem holds in the economies assumed in this study. Thanks to the theorem, any $\boldsymbol{q}_{t+1} \geq \boldsymbol{0}$ is efficient and, as we show in Subsection 3.1.4, the solution of the capitalist's problem is always consistent with that of the worker's problem in an economic system. Regarding the application of the differential game to the Pasinettian two-class model, see Chappell and Latham (1983).

3.1.3 Distribution of profits in (ILA, ILA)

In this subsection, we consider the distribution of profits between the capitalist and worker. Doing so implies specifying the condition for which θ_t must be satisfied to obtain the long-run competitive equilibrium defined in Section 2. Then, we obtain the following relationship, following **C3**:

$$\frac{(1-\theta_t)\, r_t \boldsymbol{p}^*_{t-1} \boldsymbol{A}^* \left((1-\theta_t)\, \boldsymbol{x}_t\right)}{\rho_c\, (1-\theta_t)\, r_t \boldsymbol{p}^*_{t-1} \boldsymbol{A}^* \left((1-\theta_t)\, \boldsymbol{x}_t\right)} = \frac{\theta_t r_t \boldsymbol{p}^*_{t-1} \boldsymbol{A}^* \boldsymbol{x}_t}{\rho_w \left(\boldsymbol{L}^* \boldsymbol{x}_t + r_t \boldsymbol{p}^*_{t-1} \boldsymbol{A}^* \left(\theta_t \boldsymbol{x}_t\right)\right)}, \text{ i.e.}$$

$$\theta_t r_t \boldsymbol{p}^*_{t-1} \boldsymbol{A}^* \boldsymbol{x}_t = \frac{\rho_w \boldsymbol{L}^* \boldsymbol{x}_t}{\rho_c - \rho_w},$$

In the steady state ($\boldsymbol{p}_{t-1} = \boldsymbol{p}_t, \boldsymbol{x}_t = (1+g)\, \boldsymbol{x}_{t-1}, \boldsymbol{c}_t = \boldsymbol{c}_{t-1}, \boldsymbol{d}_t = \boldsymbol{d}_{t-1}, r_{t-1} = r_t$, and $\theta_{t-1} = \theta_t$), we have

$$\theta r \boldsymbol{p}^* \boldsymbol{A}^* \boldsymbol{x} = \frac{\rho_w \boldsymbol{L}^* \boldsymbol{x}_t}{\rho_c - \rho_w}. \tag{12}$$

(12) implies that (10) is an indispensable condition for the existence of PE in the case of (ILA, ILA), as Pasinetti (1962, 1966a, 1974) asserts.

3.1.4 The long-run competitive equilibrium in (ILA, ILA)

Based on (6), (9), (11), and (12), the long-run competitive equilibrium in this case is obtained by the following system for the given rate of profit $r \in [0, R)$:

$$\left\{ \begin{array}{c} \boldsymbol{p}^* \leqq (1+r)\, \boldsymbol{p}^* \boldsymbol{A}^* + \boldsymbol{L}^*, \\ \boldsymbol{p}^* \boldsymbol{x} = (1+r) \boldsymbol{p}^* \boldsymbol{A}^* \boldsymbol{x} + \boldsymbol{L}^* \boldsymbol{x}, \\ \boldsymbol{x} \geqq (1+g)\, \boldsymbol{A}^* \boldsymbol{x} + \boldsymbol{c} + \boldsymbol{d}, \\ \boldsymbol{p}^* \boldsymbol{x} = (1+g)\, \boldsymbol{p}^* \boldsymbol{A}^* \boldsymbol{x} + \boldsymbol{p}^* \boldsymbol{c} + \boldsymbol{p}^* \boldsymbol{d}, \\ \boldsymbol{p}^* \boldsymbol{x} > 0 \\ \boldsymbol{x} \geq \boldsymbol{0}, \end{array} \right. \tag{13}$$

where

$$\left\{ \begin{array}{l} c_1 = \frac{1-\rho_c}{p_1} \left[\max \left(r \boldsymbol{p}^* \boldsymbol{A}^* \boldsymbol{x} - \frac{\rho_w}{\rho_c - \rho_w} \boldsymbol{L}^* \boldsymbol{x}, 0 \right) \right], \\ d_1 = \frac{1-\rho_w}{p_1} \left[\boldsymbol{L}^* \boldsymbol{x} + \min \left(\frac{\rho_w}{\rho_c - \rho_w} \boldsymbol{L}^* \boldsymbol{x}, r \boldsymbol{p}^* \boldsymbol{A}^* \boldsymbol{x} \right) \right]. \end{array} \right. \tag{14}$$

From the fourth equation of (13) and (14), the following relation holds when PE is obtained:

$$\begin{aligned}\boldsymbol{p}^*\boldsymbol{x} &= (1+g)\,\boldsymbol{p}^*\boldsymbol{A}^*\boldsymbol{x} + (1-\rho_c)\left[\max\left(r\boldsymbol{p}^*\boldsymbol{A}^*\boldsymbol{x} - \frac{\rho_w}{\rho_c-\rho_w}\boldsymbol{L}^*\boldsymbol{x}, 0\right)\right] \\ &\quad + (1-\rho_w)\left[\boldsymbol{L}^*\boldsymbol{x} + \min\left(\frac{\rho_w}{\rho_c-\rho_w}\boldsymbol{L}^*\boldsymbol{x}, r\boldsymbol{p}^*\boldsymbol{A}^*\boldsymbol{x}\right)\right].\end{aligned}$$

For the rate of profit such that $r\boldsymbol{p}^*\boldsymbol{A}^*\boldsymbol{x} > \frac{\rho_w}{\rho_c-\rho_w}\boldsymbol{L}^*\boldsymbol{x}$, the capitalist and worker co-exist, since the capitalist's income is kept positive.Therefore, we obtain

$$\begin{aligned}\boldsymbol{p}^*\boldsymbol{x} &= (1+g)\,\boldsymbol{p}^*\boldsymbol{A}^*\boldsymbol{x} + (1-\rho_c)\left(r\boldsymbol{p}^*\boldsymbol{A}^*\boldsymbol{x} - \frac{\rho_w}{\rho_c-\rho_w}\boldsymbol{L}^*\boldsymbol{x}\right) + (1-\rho_w)\left(\boldsymbol{L}^*\boldsymbol{x} + \frac{\rho_w}{\rho_c-\rho_w}\boldsymbol{L}^*\boldsymbol{x}\right) \\ &= \{1+g+(1-\rho_c)\,r\}\,\boldsymbol{p}^*\boldsymbol{A}^*\boldsymbol{x} + \boldsymbol{L}^*\boldsymbol{x}.\end{aligned}$$

By comparing it with the second equation of (13), we obtain

$$g = \rho_c r.$$

Similarly, for the rate of profit such that $r\boldsymbol{p}^*\boldsymbol{A}^*\boldsymbol{x} \leqq \frac{\rho_w}{\rho_c-\rho_w}\boldsymbol{L}^*\boldsymbol{x}$, as pointed out in Section 2, the capitalist disappears, since her income is zero. Thus, all incomes are distributed to the worker. Therefore, we obtain

$$\begin{aligned}\boldsymbol{p}^*\boldsymbol{x} &= (1+g)\,\boldsymbol{p}^*\boldsymbol{A}^*\boldsymbol{x} + (1-\rho_w)\,[\boldsymbol{L}^*\boldsymbol{x} + r\boldsymbol{p}^*\boldsymbol{A}^*\boldsymbol{x}] \\ &= \left[1 + \left\{g + (1-\rho_w)\,r - \frac{\rho_w}{\boldsymbol{p}^*\boldsymbol{A}^*\boldsymbol{x}}\boldsymbol{L}^*\boldsymbol{x}\right\}\right]\boldsymbol{p}^*\boldsymbol{A}^*\boldsymbol{x} + \boldsymbol{L}^*\boldsymbol{x}.\end{aligned}$$

Therefore, we obtain

$$g = \rho_w\left(r + \frac{\boldsymbol{L}^*\boldsymbol{x}}{\boldsymbol{p}^*\boldsymbol{A}^*\boldsymbol{x}}\right) = \rho_w\frac{\boldsymbol{p}^*\,(\boldsymbol{I}-\boldsymbol{A}^*)\,\boldsymbol{x}}{\boldsymbol{p}^*\boldsymbol{A}^*\boldsymbol{x}}.$$

By the same method as that of Hosoda (1989), we prove the existence of the long-run competitive activity vector $\mathbf{0} \leq \boldsymbol{x}^* \in K$. Therefore, we obtain the rate of economic growth, satisfying (13) and (14), as a function of the rate of profit, as follows:

$$g\,(r) = \begin{cases} \rho_c r & \text{if } r\boldsymbol{p}^*\boldsymbol{A}^*\boldsymbol{x}^* > \frac{\rho_w}{\rho_c-\rho_w} \text{ (the PE)}, \\ \rho_w\left(r + \frac{1}{\boldsymbol{p}^*\boldsymbol{A}^*\boldsymbol{x}^*}\right) = \rho_w\frac{\boldsymbol{p}^*(\boldsymbol{I}-\boldsymbol{A}^*)\boldsymbol{x}^*}{\boldsymbol{p}^*\boldsymbol{A}^*\boldsymbol{x}^*} & \text{if } r\boldsymbol{p}^*\boldsymbol{A}^*\boldsymbol{x}^* \leqq \frac{\rho_w}{\rho_c-\rho_w} \text{ (the DE)}. \end{cases} \tag{15}$$

Note that the activity level is normalised as $\boldsymbol{L}^*\boldsymbol{x}^* = 1$.

(15) is the exactly same result as that of Pasinetti (1962, 1974), Samuelson and Modigliani (1966a, 1966b), Hosoda (1989), and Morishima (1969). Then, it shows that the PE is characterised by the *Cambridge equation*, regardless of the long-run competitive equilibrium technique (i.e. irrespective of the elements of $\boldsymbol{A}^*$ and $\boldsymbol{L}^*$; note that the elements vary, depending on the level of the rate of profit). The rate of economic growth is determined by the worker's

saving rate multiplied by the ratio of net output to capital in DE, which implies that the relationship between the rates of economic growth and profit depends on the technology and the worker's saving rate.

Furthermore, just like in Hosoda (1989), we have

$$\rho_c r > \rho_w \left(r + \frac{1}{\boldsymbol{p}^* \boldsymbol{A}^* \boldsymbol{x}^*} \right) \qquad \text{in the PE, and}$$

$$\rho_c r \leqq \rho_w \left(r + \frac{1}{\boldsymbol{p}^* \boldsymbol{A}^* \boldsymbol{x}^*} \right) \qquad \text{in the DE.}$$

The result implies that we obtain an equilibrium with a higher rate of economic growth for $\forall r \in [0, R)$.

3.2 Optimal behaviour in the case of (capitalist, worker) = (ILA, OLG)

In this subsection, we examine the case in which the combination of the microfoundation of the capitalist and worker is represented by (capitalist, worker) = (ILA, OLG), and analyse the properties of the equilibria obtained in the case.

3.2.1 Optimal behaviour of the capitalist in (ILA, OLG)

The capitalist's behaviour is essentially the same as in the previous case. Following the notation used in the previous subsection, the problem addressed by the capitalist is given as follows:

$$\max_{c_{t,1}, \boldsymbol{x}_{t+1}} U^c \equiv (1 - \rho_c) \sum_{t=1}^{\infty} \rho_c^{t-1} \ln c_{t,1}$$

$$\text{s.t. } \boldsymbol{p}_t \boldsymbol{E} \boldsymbol{q}_{t+1} + \boldsymbol{p}_t \boldsymbol{c}_t \leqq r_t \boldsymbol{p}_{t-1} \boldsymbol{A} \left((1 - \theta_t) \boldsymbol{q}_t \right),$$

$$\text{given } \{r_t\}_{t=1,2,\cdots}, \{\theta_t\}_{t=1,2,\cdots}, \ \boldsymbol{p}_0 \geq \boldsymbol{0}.$$

The solution is obtained as follows:

$$\boldsymbol{p}_t^* \boldsymbol{c}_t = (1 - \rho_c) r_t \boldsymbol{p}_{t-1}^* \boldsymbol{A}^* \left((1 - \theta_t) \boldsymbol{x}_t \right), \text{ i.e.} \tag{16}$$

$$c_{t,1} = \frac{(1 - \rho_c)(1 - \theta_t) r_t \boldsymbol{p}_{t-1}^* \boldsymbol{A}^* \boldsymbol{x}_t}{p_{t,1}^*}.$$

As in Subsection 3.1.1, (16) implies that the capitalist consumes commodity 1 proportionally to her income $r_t \boldsymbol{p}_{t-1}^* \boldsymbol{A}^* \left((1 - \theta_t) \boldsymbol{x}_t \right)$, and thus, the capitalist's saving rate is ρ_c.

3.2.2 Optimal behaviour of the worker in (ILA, OLG)

Since the worker's behaviour is characterised by the OLG model, he lives for two periods. The worker, who is born at time t, supplies the labour force and obtains wage $\boldsymbol{L}\boldsymbol{q}_t$ (note that the wage rate is normalised as $w_t = 1$) at time t. This is his young age. At this age, he must solve the problem of allocating the wage between consumption and saving, and gains his livelihood at time $t+1$ by this saving. This time represents his old age. Therefore, the budget constraint for the young age of the worker born at time t is written as $\boldsymbol{p}_t\boldsymbol{d}_t^b + S_t \leqq \boldsymbol{L}\boldsymbol{q}_t$, where $\boldsymbol{d}_t^b \equiv \left(d_{t,1}^b, 0, \cdots, 0\right) \in \mathbb{R}_+^n$ and S_t denote the consumption vector for the young age of the worker born at time t and the saving, respectively. Moreover, the budget constraint for the old age of the worker born at time t is given by $\boldsymbol{p}_{t+1}\boldsymbol{d}_t^a \leqq (1 + r_{t+1})S_t$, where $\boldsymbol{d}_t^a \equiv \left(d_{t,1}^a, 0, \cdots, 0\right) \in \mathbb{R}_+^n$ denotes the consumption vector for the old age of the worker born at time t. By combining these budget constraints, we obtain the worker's intertemporal budget constraint.

Therefore, the worker born at time t solves the following problem:

$$\begin{aligned} \max_{d_{t,1}^b, d_{t,1}^a} \; & (1-\rho_w)\ln d_{t,1}^b + \rho_w \ln d_{t,1}^a \\ \text{s.t.} \; & \boldsymbol{p}_t\boldsymbol{d}_t^b + \frac{1}{1+r_{t+1}}\boldsymbol{p}_{t+1}\boldsymbol{d}_t^a \leqq \boldsymbol{L}\boldsymbol{q}_t, \\ \text{given} \; & \{r_t\}_{t=1,2,\cdots}, \{\boldsymbol{q}_t\}_{t=1,2\cdots}, \boldsymbol{p}_0 \geq \boldsymbol{0}. \end{aligned}$$

The solution is given as follows:

$$\boldsymbol{p}_t^*\boldsymbol{d}_t^b = (1-\rho_w)\,\boldsymbol{L}^*\boldsymbol{x}_t, \quad \text{i.e. } d_{t,1}^b = \frac{(1-\rho_w)\,\boldsymbol{L}^*\boldsymbol{x}_t}{p_{t,1}^*}, \tag{17}$$

$$\boldsymbol{p}_{t+1}^*\boldsymbol{d}_t^a = \rho_w\,(1+r_{t+1})\,\boldsymbol{L}^*\boldsymbol{x}_t, \quad \text{i.e. } d_{t,1}^a = \frac{\rho_w\,(1+r_{t+1})\,\boldsymbol{L}^*\boldsymbol{x}_t}{p_{t+1,1}^*}. \tag{18}$$

(17) implies that the worker in young age consumes consumption 1 proportionally to his income $\boldsymbol{L}^*\boldsymbol{x}_t$. Therefore, the worker's saving rate is ρ_w. (18) means that the worker in old age maintains his livelihood by dissaving (i.e. using his saving).

3.2.3 Distribution of profits in (ILA, OLG)

The property of the OLG model means that the profits distributed to the worker at time t must equal the income that the worker, born at time $t-1$, receives in old age (i.e. $\rho_w\,(1+r_t)\,\boldsymbol{L}^*\boldsymbol{x}_{t-1} = \theta_t r_t \boldsymbol{p}_{t-1}^*\boldsymbol{A}^*\boldsymbol{x}_t$). Following **C3**, we obtain the distribution of profits between the capitalist and worker as follows:

$$\frac{(1-\theta_t)\, r_t \boldsymbol{p}_{t-1}^* \boldsymbol{A}^* \boldsymbol{x}_t}{\rho_c (1-\theta_t)\, r_t \boldsymbol{p}_{t-1}^* \boldsymbol{A}^* \boldsymbol{x}_t} = \frac{\theta_t r_t \boldsymbol{p}_{t-1}^* \boldsymbol{A}^* \boldsymbol{x}_t}{\rho_w \boldsymbol{L}^* \boldsymbol{x}_t}, \text{ i.e.}$$

$$\theta_t r_t \boldsymbol{p}_{t-1}^* \boldsymbol{A}^* \boldsymbol{x}_t = \frac{\rho_w}{\rho_c} \boldsymbol{L}^* \boldsymbol{x}_t.$$

As argued in detail in Subsection 3.4, (18) implies that we need not assume $\rho_w < \rho_c$ to ensure the existence of PE in the case of (ILA, OLG), in contrast to (12). We obtain (19) in the steady state

$$\theta r \boldsymbol{p}^* \boldsymbol{A}^* \boldsymbol{x} = \frac{\rho_w}{\rho_c} \boldsymbol{L}^* \boldsymbol{x}. \tag{19}$$

3.3 The long-run competitive equilibrium in (ILA, OLG)

The property of the OLG model means that workers in young age and old age co-exist within a period of time. Therefore, based on (16)–(19), the steady-state consumption vectors of the capitalist and workers are

$$\begin{cases} c_1 = \frac{1-\rho_c}{p_1^*} \max\left[r\boldsymbol{p}^*\boldsymbol{A}^*\boldsymbol{x} - \frac{\rho_w}{\rho_c}\boldsymbol{L}^*\boldsymbol{x}, 0\right] \\ d_1 = d_1^a + d_1^b = \frac{1}{p_1^*}\left[\min\left(\frac{\rho_w}{\rho_c}\boldsymbol{L}^*\boldsymbol{x}, r\boldsymbol{p}^*\boldsymbol{A}^*\boldsymbol{x}\right) + (1-\rho_w)\,\boldsymbol{L}^*\boldsymbol{x}\right] \end{cases} \tag{20}$$

respectively. Then, the long-run competitive equilibrium can be defined by (13) for the given rate of profit $r \in [0, R)$. Similarly, in Subsection 3.1.4, we obtain the following from the fourth equation of (13) and (20):

$$\begin{aligned} \boldsymbol{p}^*\boldsymbol{x} &= (1+g)\,\boldsymbol{p}^*\boldsymbol{A}^*\boldsymbol{x} + (1-\rho_c)\max\left[r\boldsymbol{p}^*\boldsymbol{A}^*\boldsymbol{x} - \frac{\rho_w}{\rho_c}\boldsymbol{L}^*\boldsymbol{x}, 0\right] \\ &\quad + \min\left(\frac{\rho_w}{\rho_c}\boldsymbol{L}^*\boldsymbol{x}_t, r\boldsymbol{p}^*\boldsymbol{A}^*\boldsymbol{x}\right) + (1-\rho_w)\,\boldsymbol{L}^*\boldsymbol{x}. \end{aligned}$$

For the rate of profit such that $r\boldsymbol{p}^*\boldsymbol{A}^*\boldsymbol{x} > \frac{\rho_w}{\rho_c}\boldsymbol{L}^*\boldsymbol{x}$, it can be transformed as follows:

$$\begin{aligned} \boldsymbol{p}^*\boldsymbol{x} &= (1+g)\,\boldsymbol{p}^*\boldsymbol{A}^*\boldsymbol{x} + (1-\rho_c)\left(r\boldsymbol{p}^*\boldsymbol{A}^*x - \frac{\rho_w}{\rho_c}\boldsymbol{L}^*\boldsymbol{x}_t\right) + \frac{\rho_w}{\rho_c}\boldsymbol{L}^*\boldsymbol{x}_t + (1-\rho_w)\,\boldsymbol{L}^*\boldsymbol{x} \\ &= \{1+g+(1-\rho_c)\,r\}\,\boldsymbol{p}^*\boldsymbol{A}^*\boldsymbol{x} + \boldsymbol{L}^*\boldsymbol{x}. \end{aligned}$$

By comparing it with the second equation of (13), we obtain

$$g = \rho_c r.$$

For the rate of profit such that $r\boldsymbol{p}^*\boldsymbol{A}^*\boldsymbol{x} \leqq \frac{\rho_w}{\rho_c}\boldsymbol{L}^*\boldsymbol{x}$, we obtain

$$\begin{aligned}\boldsymbol{p}^*\boldsymbol{x} &= (1+g)\,\boldsymbol{p}^*\boldsymbol{A}^*\boldsymbol{x} + r\boldsymbol{p}^*\boldsymbol{A}^*\boldsymbol{x} + (1-\rho_w)\,\boldsymbol{L}^*\boldsymbol{x} \\ &= \left(1+g+r-\frac{\rho_w\boldsymbol{L}^*\boldsymbol{x}}{\boldsymbol{p}^*\boldsymbol{A}^*\boldsymbol{x}}\right)\boldsymbol{p}^*\boldsymbol{A}^*\boldsymbol{x} + \boldsymbol{L}^*\boldsymbol{x}.\end{aligned}$$

By comparing it with the second equation of (13), we obtain

$$g = \rho_w \frac{\boldsymbol{L}^*\boldsymbol{x}}{\boldsymbol{p}^*\boldsymbol{A}^*\boldsymbol{x}}.$$

Using the same method as that of Hosoda (1989), we can prove the existence of the long-run competitive activity vector $\boldsymbol{x}^* \geq \mathbf{0}$, together with $\boldsymbol{p}^* \geq \mathbf{0}$, in set K. Therefore, we obtain the rate of economic growth, satisfying (13) and (20), as a function with respect to the rate of profit, as follows:

$$g(r) = \begin{cases} \rho_c r \quad \text{if } r\boldsymbol{p}^*\boldsymbol{A}^*\boldsymbol{x}^* > \frac{\rho_w}{\rho_c} \text{ (the PE)}, \\ \frac{\rho_w}{\boldsymbol{p}^*\boldsymbol{A}^*\boldsymbol{x}^*} \quad \text{if } r\boldsymbol{p}^*\boldsymbol{A}^*\boldsymbol{x}^* \leqq \frac{\rho_w}{\rho_c} \text{ (the DE)}. \end{cases} \tag{21}$$

Note that the activity level is normalised as $\boldsymbol{L}^*\boldsymbol{x}^* = 1$.

(21) demonstrates that the PE is substantially characterised by the *Cambridge equation*, whichever technique is cost minimising, and the rate of economic growth in the DE is determined by the worker's saving rate and the ratio of labour to capital in the case of (ILA, OLG). As is the case of (ILA, ILA), the relationship between the rates of economic growth and profit depends on the technology and the worker's saving rate.

Moreover, when the PE holds in this case ($r\boldsymbol{p}^*\boldsymbol{A}^*\boldsymbol{x}^* > \frac{\rho_w}{\rho_c}$), we have

$$\rho_c r > \rho_w \frac{1}{\boldsymbol{p}^*\boldsymbol{A}^*\boldsymbol{x}^*}.$$

Similarly, when the DE holds ($r\boldsymbol{p}^*\boldsymbol{A}^*\boldsymbol{x}^* \leqq \frac{\rho_w}{\rho_c}$), we have:

$$\rho_c r \leqq \frac{\rho_w}{\boldsymbol{p}^*\boldsymbol{A}^*\boldsymbol{x}^*}.$$

The result implies that we obtain an equilibrium with a higher rate of economic growth for $\forall r \in [0, R)$, as in Subsection 3.1.4.

3.4 Short remarks of Section 3

As shown by (15) and (21), the relationship between the rates of economic growth and profit in PE, in which the capitalist and worker co-exist in the steady state, is given by $g = \rho_c r$ irrespective of the microfoundation of the capitalist and worker. Thus, the relationship is independent of the long-run competitive equilibrium technique and has substantially the same implication as the *Cambridge equation*. In other words, the relationship is independent of the technology and the worker's saving rate, and is determined solely by the capitalist's saving rate. Since the choice of the assumptions on individuals' behaviour can be regarded as the *institution* in the sense of Pasinetti (2007), our results confirm Pasinetti's (2012, p.

1440) statement that 'the *Cambridge equation* is independent of the institutional set-up of the society that is considered'.

On the contrary, the relationship between the rate of economic growth and that of profit in DE, in which the capitalist disappears and all the stock of capital is owned by the worker, depends entirely on the assumptions of the microfoundation. Whereas (15) demonstrates that the rate of economic growth is determined by the worker's saving rate multiplied by the ratio of net output to capital in the case of (ILA, ILA), (21) means that it is determined by the worker's saving rate multiplied by the ratio of labour to capital in the case of (ILA, OLG). In both cases, the relationship is dependent on the technology, which is in contrast to the PE. Based on Pasinetti (1966a, 1974), the dependence of the relationship between the rates of economic growth and profit on the technology in the DE provides Samuelson and Modigliani (1966a, 1966b) with the opportunity to resurrect the neo-classical well-behaved production function.

The difference in the relationship in the DE is solely attributed to that of the worker's behaviour, which differentiates the worker's saving functions in the case of (ILA, ILA) and (ILA, OLG). Pasinetti (1962, p. 270) criticises the Kaldorian saving function ($S_t = s_c P_t + s_w W_t$, where $s_c > s_w$), as 'a logical slip' and argues that the *Cambridge equation* is obtained in Kaldor (1956) only under the restrictive assumption of $s_w = 0$. If the worker saves a part of his income, he must be allowed to own it and thus, receives the income from not only wage but also capital. Then, Pasinetti (1962) proposes a new saving function alternative to the Kaldorian, as shown in Section 2: $S_t = s_c P_t^c + s_w (W_t + P_c^w)$, where $s_c > s_w$ must be assumed. As implied by (9) and (11), Pasinetti's total saving function corresponds to the case of (ILA, ILA). As implied by (11), the worker in this case saves from not only his wage but also the profit distributed to him.

It is obvious that if a worker saves, then he receives the profit. However, it does not necessarily mean that the profit is saved. This is demonstrated by the case of (ILA, OLG). Although the worker certainly receives not only wage but also profit in this case, he saves only the wage part and consumes the profit entirely. Following the notation used in Section 1, the total saving function in this case can be written as $S_t = s_c P_t^c + s_w W_t$. As suggested by (19), we should pay attention to the result that (10) – whether the capitalist's saving rate is greater than the worker's – is dispensable for the existence of the PE in the case of (ILA, OLG). We find that the PE exists even when the worker's saving rate is higher than the capitalist's in this case, as indicated by the numerical example in Section 4. Caggetti and De Nardi (2008) and De Nardi (2015) imply that people whose behaviour follows the OLG model cannot be thought of as an unrealistic or extreme case, as mentioned in Section 1, since the introduction of their behaviour into the models leads to a better fit with the data.

Concerning the Kaldorian saving function, Samuelson and Modigliani (1966a) suggest reinterpreting the case in which the worker saves a part of earned profit *as if* he were a capitalist, and thus, they argue that there is no 'logical slip'. Pasinetti (1983) criticises Samuelson and Modigliani's (1966a) re-interpretation for being 'incompatible with steady growth'. Here, 'steady growth' means PE (i.e. the growth path on which both the capitalist and worker co-exist). The case of (ILA, OLG) is the combination of the microfoundation under which there is no 'logical slip' and the compatibility with the 'steady growth' is ensured, irrespective of whether the capitalist' saving rate is greater than the worker's, if **C3** holds.

4 Type of Equilibria and Choice of Technique

In the previous sections, we concentrate on the relationship between the rates of economic growth and profit obtained in PE and DE. As pointed out in Section 1, the advantage of our model is that the switch of the type of equilibria can be examined simultaneously with the choice of technique (the occurrence of paradoxes in capital theory). We analyse them by using the numerical examples, and then derive the propositions from our model.

First, we confirm the concepts used in this section. Let $\boldsymbol{b} \in \mathbb{R}_+^n$ be a bundle of commodities adopted as the numéraire. For the rate of profit such that $r \in [0, R_\iota)$, where R_ι denotes the maximum rate of profit attainable by technique $\iota \in \Delta$, the wage curve of technique ι measured by $\boldsymbol{b}$ can be defined as follows:

$$w_\iota(r) = \frac{1}{\boldsymbol{L}_\iota \left[\boldsymbol{I} - (1+r)\boldsymbol{A}_\iota\right]^{-1} \boldsymbol{b}}.$$

$w_\iota(r)$ is a function of r that corresponds to $(\boldsymbol{A}_\iota, \boldsymbol{L}_\iota)$ for $\iota \in \Delta$. Furthermore, we define the factor price frontier $\omega(r)$ as follows:

$$\omega(r) \equiv \max_{\iota \in \Delta} w_\iota(r).$$

In other words, the factor price frontier is the outermost envelope of the wage curves, which implies that the cost-minimising technique for the given rate of profit is that by which the highest wage rate can be offered (Kurz and Salvadori, 1995, p. 142).

4.1 Numerical Example

We consider the following numerical example:

Example 1: We suppose a two-class economy with two commodities. Commodity 1 has only one process: $[\boldsymbol{a}_1, L_1] = \left[\begin{pmatrix} 2/5 \\ 2 \end{pmatrix}, 1\right]$ and commodity 2 has two alternative processes α and β: $[\boldsymbol{a}_{2\alpha}, L_{2\alpha}] = \left[\begin{pmatrix} 1/40 \\ 1/10 \end{pmatrix}, 1\right]$ and $[\boldsymbol{a}_{2\beta}, L_{2\beta}] = \left[\begin{pmatrix} 0.0001 \\ 113/232 \end{pmatrix}, \frac{275}{464}\right]$.[8] The technology available to the whole economy are given as follows:

$$\boldsymbol{A} = \begin{pmatrix} 2/5 & 1/40 & 0.0001 \\ 2 & 1/10 & 113/232 \end{pmatrix}, \boldsymbol{L} = \begin{pmatrix} 1 & 1 & \frac{275}{464} \end{pmatrix}.$$

Therefore, we have

$$[\boldsymbol{A}_\alpha, \boldsymbol{L}_\alpha] = \left[\begin{pmatrix} 2/5 & 1/40 \\ 2 & 1/10 \end{pmatrix}, (1, 1)\right], \quad [\boldsymbol{A}_\beta, \boldsymbol{L}_\beta] = \left[\begin{pmatrix} 2/5 & 0.0001 \\ 2 & 113/232 \end{pmatrix}, \left(1, \frac{275}{464}\right)\right].$$

Here, the set $\Delta = \{\alpha, \beta\}$ is defined. We consider the wage curves of technique ι for $\iota \in \Delta$ obtained by adopting a bundle of commodities $\boldsymbol{b} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ as the numéraire. Figure 1 depicts the wage curves and thus, the factor price frontier as their outermost envelope.

[8] The numerical example is from Vienneau (2005).

Insert Figure 1 here.

The intersection of the wage curves, called the switch points, are given by $r \approx 0.197$, 0.807. As the long-run competitive equilibrium technique (i.e. cost-minimising technique for given rate of profit), β is chosen at $r \in [0, 0.197)$, α is chosen at $r \in (0.197, 0.807)$, and β is chosen again at $r \in (0.807, 1.04)$, where $R_\beta = 1.04$.[9] Figure 1 shows that both reswitching of technique and reverse capital deepening occur.[10] Since technique β, which is chosen in $r \in [0, 0.197)$, is chosen again in $r \in (0.807, 1.04)$, the reswitching of technique occurs. Since the switch of technique from α to β at $r \approx 0.8007$ implies that technique with higher capital intensity is chosen despite the rise of the rate of profit, reverse capital deepening occurs. In the following part, we assume $\rho_c = 0.7$ and $\rho_w = 0.3$, unless otherwise stated.

4.1.1 Relationship between the rate of economic growth and that of profit in the case of (ILA, ILA)

Under the above assumptions, we obtain the relationship between the rates of economic growth and profit, given by (15). Figure 2 shows the relationship under the technology given in Example 1.

Insert Figure 2 here.

In Figure 2, g_α and g_β denote the rate of economic growth as a function of the rate of profit in the DE when α and β, respectively, are activated as the long-run competitive equilibrium technique. The rate of economic growth as a function of the rate of profit in the PE is indicated by $g = 0.7r$, whichever is the long-run competitive equilibrium technique.

As confirmed in Figure 1, technique β is the long-run competitive equilibrium technique for $r \in [0, 0.197)$ and thus, is chosen in the interval of the rate of profit. Note that $0.7r < 0.3 \left(r + \frac{1}{\boldsymbol{p}^* \boldsymbol{A}^* \boldsymbol{x}^*} \right) \equiv g_\beta (r)$, where $\boldsymbol{A}^* = \boldsymbol{A}_\beta$, holds for $r \in [0, 0.197)$. Since we have a higher rate of economic growth in the steady state, as argued in Section 3.1.4, the DE is obtained under technique β in the interval of the rate of profit. For $r \in (0.197, 0.807)$, as shown in Figure 1, the long-run competitive equilibrium technique is α. Because of $g_\alpha (r) \equiv 0.3 \left(r + \frac{1}{\boldsymbol{p}^* \boldsymbol{A}^* \boldsymbol{x}^*} \right)$, where $\boldsymbol{A}^* = \boldsymbol{A}_\alpha$, the intersection between $g = 0.7r$ and $g = g_\alpha (r)$ is given by $r \approx 0.325$. Then, $0.7r \leqq g_\alpha (r)$ holds for $r \in (0.197, 0.325]$. This result implies

[9] Note that each commodity has the same price at the switch points irrespective of whether it is produced by any technique or any convex combination of techniques. See Kurz and Salvadori (1995, p. 142) and Pasinetti (1977, pp. 158–159).

[10] These phenomena are inconsistent with the principle of marginal productivity of capital. According to the principle, there is a one-to-one correspondence between the rate of profit and the cost-minimising technique. The reswitching of technique means that one technique, which is chosen as the cost-minimising technique at the rate of profit, is also chosen at other rate of profit. Moreover, according to the principle, the capital intensity (value of capital per worker) is a monotonically decreasing function of the rate of profit. Reverse capital deepening (or capital reversing) means that the increase in the rate of profit raises the capital intensity. See, for example, Kurz and Salvadori (1995, pp. 147–149) and Pasinetti (1977, pp. 169–173) for details.

that the DE is obtained under technique α for $r \in (0.197, 0.325]$. For $r \in (0.325, 0.807)$, on the contrary, $0.7r > g_\alpha(r)$ holds, implying that the PE is obtained under technique α in the interval of the rate of profit. Furthermore, for $r \in (0.807, 1.04)$, the long-run competitive equilibrium technique is β again. Recall that reswitching of technique and reverse capital deepening take place in the switching technique from α to β at $r \approx 0.807$. For $r \in (0.807, 1.04)$, $0.7r > g_\beta(r)$ holds. This result means that the PE is obtained for in the interval of the rate of profit. Note that the relationship between the rate of economic growth and that of profit in the PE, represented by $g = 0.7r$, is unchanged even though the long-run competitive equilibrium technique changes from α to β.

Since the cost of production is always minimised at the switch points regardless of which technique or convex combination of existing techniques is used, as mentioned in footnote 9, the long-run competitive equilibrium is consistent with the multiple rates of economic growth at the switch points. However, it would be reasonable to suppose that the highest rate of economic growth is preferred. Then, the DE is obtained under technique α at $r \approx 0.197$ and the PE at $r \approx 0.807$.

Let $\sigma_\iota(r) \equiv \frac{g_\iota(r)}{\rho_w}$ for $\iota \in \Delta$. From the definition of $g = g_\iota(r)$, $\sigma_\iota(r)$ denotes the ratio of net output to capital when $\iota \in \Delta$ is the long-run competitive equilibrium technique at $r \in [0, R_\iota)$. Under the technology given in Example 1, $0.268 \approx g_\alpha(0.807) < g_\beta(0.807) \approx 0.294$.[11] This result implies that $\sigma_\alpha(0.807) < \sigma_\beta(0.807)$. In other words, the ratio of net output to capital rises when the long-run competitive equilibrium technique switches from α to β at the switch point $r \approx 0.807$. The movement of $\sigma_\iota(r)$ around $r \approx 0.807$ is compatible with the principle of marginal productivity of capital. However, the movement of the capital intensity when the long-run competitive equilibrium technique switches from α to β at $r \approx 0.807$ is inconsistent with the principle, since reverse capital deepening occurs. Figure 2 demonstrates that among two indexes characterising the technique in the neo-classical production function, one movement (ratio of net output to capital) is compatible with the principle but the other (capital intensity) is not. The result cannot be obtained by the model in which capital is treated as the primary factor of production. When capital is composed of a bundle of reproducible and heterogeneous commodities, the phenomena incompatible with the principle are not necessarily considered as paradoxical.

4.1.2 Relationship between the rates of economic growth and profit in the case of (ILA, OLG)

Next, we examine the case of (ILA, OLG) under the technology given in Example 1. Figure 3 shows the relationship between the rates of economic growth and profit, given by (21):

Insert Figure 3 here.

In Figure 3, similar to Subsection 4.1.1, g_α and g_β denote the the rate of economic growth as a function of the rate of profit in the DE when α and β, respectively, are utilised as the long-run competitive equilibrium technique. Note that $g_\iota(r)$ is the worker's saving rate

[11]Note that $g = g_\alpha(r)$ and $g = g_\beta(r)$ in this case have the intersection at $r \approx 0.796$.

multiplied by the *ratio of labour to capital* (reciprocal of capital intensity) in this case. The rate of economic growth as a function of the rate of profit in the PE is given by $g = 0.7r$, which is the same function as that obtained in Subsection 4.1.1. This verifies the result obtained in (15) and (21).

The intersection between $g = g_\beta(r)$ and $g = 0.7r$ is given by $r \approx 0.191$. Therefore, the DE is obtained for $r \in [0, 0.191]$ and the PE for $r \in (0.191, 0.197)$ under technique β. The DE is obtained for $r \in [0.197, 0.223]$ and the PE is obtained for $r \in (0.223, 0.807)$ under technique α, where $r \approx 0.223$ is the intersection between $g = g_\alpha(r)$ and $g = 0.7r$. For $r \in (0.807, 1.04)$, moreover, the PE is obtained under technique β.

4.1.3 The case of $\rho_w > \rho_c$ in (ILA, OLG)

As already pointed out in Subsection 3.4, (10) is an indispensable assumption to ensure the existence of the PE in the case of (ILA, ILA) but not in the case of (ILA, OLG). In this subsection, we examine the case of $\rho_w > \rho_c$ in (ILA, OLG) by assuming the technology given in Example 1 and $\rho_w = 0.6$, $\rho_c = 0.35$. This case is unrealistic but is a logical exercise to test the working of the Pasinettian two-class model with a microfoundation of capitalist and worker.

Figure 4 shows the relationship between the rates of economic growth and profit in this case:

Insert Figure 4 here.

The figure indicates that the DE is obtained under technique β for $r \in [0, 0.197)$, and the DE is obtained under technique α for $r \in [0.197, 0.735]$, where $r \approx 0.735$ is the intersection between $g = g_\alpha(r)$ and $g = 0.35r$. Furthermore, the PE is certainly obtained under technique α for $r \in (0.735, 0.807)$ and under technique β for $r \in [0.807, 1.04)$.

4.2 Short remarks of Section 4

In Section 4, we closely examine the results obtained in Section 3 by using the numerical examples. Although these examples may be specific, they provide useful insight of the analysis of the distribution of income and capital (wealth).

From Figures 2, 3, and 4, we observe that even when techniques switch from one to another, the relationship between the rates of economic growth and profit obtained in the PE, which is given by $g = \rho_c r$, is independent of the technology and the microfoundation of the capitalist's and worker's behaviour. In addition, we observe from the figures that the relationship obtained in the DE depends on the technology and the microfoundation of agents for the given rate of profit.

Although the movement of the capital intensity around the switch point exhibits paradoxical behaviour in Figure 2, that of the ratio of net output to capital is compatible with the principle of marginal productivity of capital. This interesting result is obtained only in the model in which capital is treated as a bundle of reproducible and heterogeneous commodities.

Moreover, the phenomena inconsistent with the principle of marginal productivity of capital are not necessarily considered as a paradox.

The fundamental cause of such paradoxical phenomena arising is our assumption that capital consists of a bundle of reproducible and heterogeneous commodities and not that our model lacks a continuous and differentiable production function. Obviously, our assumption on capital is realistic. Burmeister (1981) shows that even when neo-classical well-behaved production functions are assumed, the possibility of paradoxical phenomena arising cannot be excluded if capital consists of reproducible and heterogeneous commodities. To exclude this possibility, a peculiar assumption which is unnecessary in the one-commodity model needs to be made.[12]

The rate of economic growth obtained in the PE is necessarily an increasing function of that of profit, irrespective of the microfoundation of agents and the technology. Although the relationships characterising the DE in the cases of both (ILA, ILA) and (ILA, OLG) are also an increasing function of the rate of profit in Example 1, we can easily find numerical examples in which the rate of economic growth is a decreasing function of that of profit in the DE, depending on the assumption of the microfoundation of the agents.[13] Complicated relationships between the rate of economic growth and income distribution can be obtained only by employing multi-sectoral models with capital as a bundle of reproducible and heterogeneous commodities. In fact, we know little about the movement of $g_\iota(r)$ for $\iota \in \Delta$ unless the neo-classical production function is assumed, since it depends on the property of the technology and there are complicated effects on the aggregation of income and capital when capital consists of a bundle of reproducible and heterogeneous commodities.

5 Concluding Remarks

In this study, we examine the distribution of income and capital (wealth) by a multi-sectoral two-class model with a microfoundation of capitalists and workers. The characteristic of our model is to treat capital as a bundle of reproducible and heterogeneous commodities. Clearly, capital in that sense, rather than capital as the primary factor of production, is a typical factor of production in modern capitalist economies. Our model can be considered as a summary of the Cambridge capital controversy in that it enables us to comprehensively analyse the major issues of the controversy (i.e. controversy about the *Cambridge equation* and the choice of technique) by using a single model. The aggregate macroeconomic model is used to address the former issue and the multi-sectoral model à la the Leontief–Sraffa is used to address the latter.

[12] According to Burmeister (1981), this peculiar assumption is that the so-called real Wicksell effect is negative for all feasible $r > 0$:

$$\sum_{i=1}^{n} p_i(r) \frac{\mathrm{d}k_i(r)}{\mathrm{d}r} < 0,$$

where $p_i(r)$ and $k_i(r)$ denote the relative price of commodity i and the per capita amount of the commodity necessary as the capital input, respectively. As our analysis shows, it is not necessary to assume that the inequality is likely to be satisfied.

[13] See Appendix A for a numerical example in which the rate of economic growth is a decreasing function of that of profit.

The movements of the ratio of net output to capital, the capital intensity, and the rate of profit (or wage rate) are important variables for analysis of the distribution of income and capital (wealth). According to the principle of marginal productivity of capital, the ratio of net output to capital declines when the wage rate rises. Stiglitz (2016) considers the recent rise in the US ratio of wealth to income without a substantial increase in the wage rate to be a paradox, and argues that the movement of the ratio cannot be explained by the standard neo-classical growth theory with the well-behaved production function.

We show that the movements of the ratio of net output to capital and the capital intensity when the rate of profit changes could be more complicated than those predicted by the models assuming the neo-classical well-behaved production function. This is because not only the quantities but also prices of capital change when the rate of profit changes if capital is treated as a bundle of reproducible and heterogeneous commodities. The changes in both quantities and prices generate the complicated effects on the aggregation of income and capital. If capital is treated as the primary and homogeneous factor of production, as in the neo-classical production function, then the change in the rate of profit generates only a change in the quantities. According to the lesson of the Cambridge capital controversy, the phenomena indicated by Stiglitz are not necessarily a paradox, as we show in Sections 3 and 4. The growth models assuming the neo-classical well-behaved production function are not useful for predicting what happens to the steady states of the two-class economies on which **C3** is imposed. This result implies that it is necessary to analyse the distribution of income and capital (wealth) by the models with capital as a bundle of reproducible and heterogeneous commodities.

Of course, our study leaves some open questions. We focus only on the properties of steady states. Significant remaining questions are whether an economy starting from the initial value converges toward the steady state and, if it does, what equilibria it is likely to converge toward. Examination of these questions in the two-class model with capital as a bundle of reproducible and heterogeneous commodities remains as our future research agenda.

6 Appendix A

Here, we show the numerical example in which the rate of economic growth is a decreasing function of that of profit:

Example 2:[14] We assume a two-class economy with two commodities. Commodity 1 has two alternative processes γ and δ: $[\boldsymbol{a}_{\gamma 1}, L_{\gamma 1}] = \left[\begin{pmatrix} 0 \\ 0.5 \end{pmatrix}, 1\right]$, $[\boldsymbol{a}_{\delta 1}, L_{\delta 1}] = \left[\begin{pmatrix} 0 \\ 0.125 \end{pmatrix}, 2\right]$. In addition, commodity 2 has only one process: $[\boldsymbol{a}_{2}, L_{2}] = \left[\begin{pmatrix} 0.5 \\ 0 \end{pmatrix}, 1\right]$. Therefore, the technology available in the economy as a whole is given as follows:

$$\boldsymbol{A} = \begin{pmatrix} 0 & 0 & 0.5 \\ 0.5 & 0.125 & 0 \end{pmatrix}, \boldsymbol{L} = \begin{pmatrix} 1 & 2 & 1 \end{pmatrix}.$$

[14]The example is from Hosoda (1989).

Then, we have

$$[\boldsymbol{A}_\gamma, \boldsymbol{L}_\gamma] = \left[\begin{pmatrix} 0 & 0.5 \\ 0.5 & 0 \end{pmatrix}, (1,1)\right], \quad [\boldsymbol{A}_\delta, \boldsymbol{L}_\delta] = \left[\begin{pmatrix} 0 & 0.5 \\ 0.125 & 0 \end{pmatrix}, (2,1)\right].$$

Here, the set $\Delta = \{\gamma, \delta\}$ can be defined. Figure 5 depicts the wage curves of technique $\iota \in \Delta$ and the factor price frontier. It demonstrates that neither reswitching of technique nor reverse capital deepening occur: the technique switches from γ to δ at $r \approx 0.143$, and technique δ, which has lower capital intensity than technique γ, is chosen for $r \in [0.143, 3]$. Note that the wage curve of technique γ is linear, since the so-called organic composition of capital is uniform in both sectors.

Insert Figure 5 here.

We assume that $\rho_c = 0.7$ and $\rho_w = 0.3$ as before. From (15) and (21), we obtain the rate of economic growth as the function of the rate of profit in the case of (ILA, ILA) and (ILA, OLG). Figures 6 and 7 show the relationship between the rate of economic growth and that of profit in the former case and that in the latter case, respectively. Note that the price is normalised by $w = 1$ and the activity level by $\boldsymbol{L}^* \boldsymbol{x}^* = 1$ for $\forall r \in [0, R)$.[15] Figure 6 (ILA, ILA) shows that the DE obtained under technique γ is indicated by g_γ for $r \in [0, 0.143)$; the DE obtained under technique δ is indicated by g_δ for $r \in [0.143, 0.917]$, where $r \approx 0.917$ is the intersection between $g = 0.7r$ and $g = g_\delta(r)$; and the PE is obtained under technique δ for $r \in (0.917, 3)$, where $r = 3$ is the maximum rate of profit obtainable under technique δ. Figure 7 (ILA, OLG) shows that the DE is obtained under technique γ for $r \in [0, 0.143)$; and the DE is obtained for $r \in [0.143, 0.994]$, where $r \approx 0.994$ is the intersection between $g = 0.7r$ and $g = g_\delta(r)$; and the PE is obtained under technique δ for $r \in (0.994, 3)$.

Insert Figures 6 & 7 here.

Note that g_γ in Figure 6 is constant independently of the level of r, since the ratio of net output to capital is constant irrespective of the level of r if the so-called organic composition of capital is uniform in all sectors (Pasinetti, 1977, p. 86). g_δ in Figure 6 is not a monotonic function; the rate of economic growth is minimised at $r \approx 1.067$. Therefore, we obtain the rate of economic growth as a decreasing function of the rate of profit for $r \in [0.143, 0.917]$. In Figure 7, both g_γ and g_δ are decreasing functions of the rate of profit.

[15] Hosoda (1989) also draws the figure in the case of (ILA, ILA) based on the same numerical example. Although function (II) of Figure 6 in his study corresponds to g_δ in our Figure 6, the form of his function (II) is different from that of our g_δ. We assume that this is because he does not normalise the prices by $w = 1$, unlike in our study.

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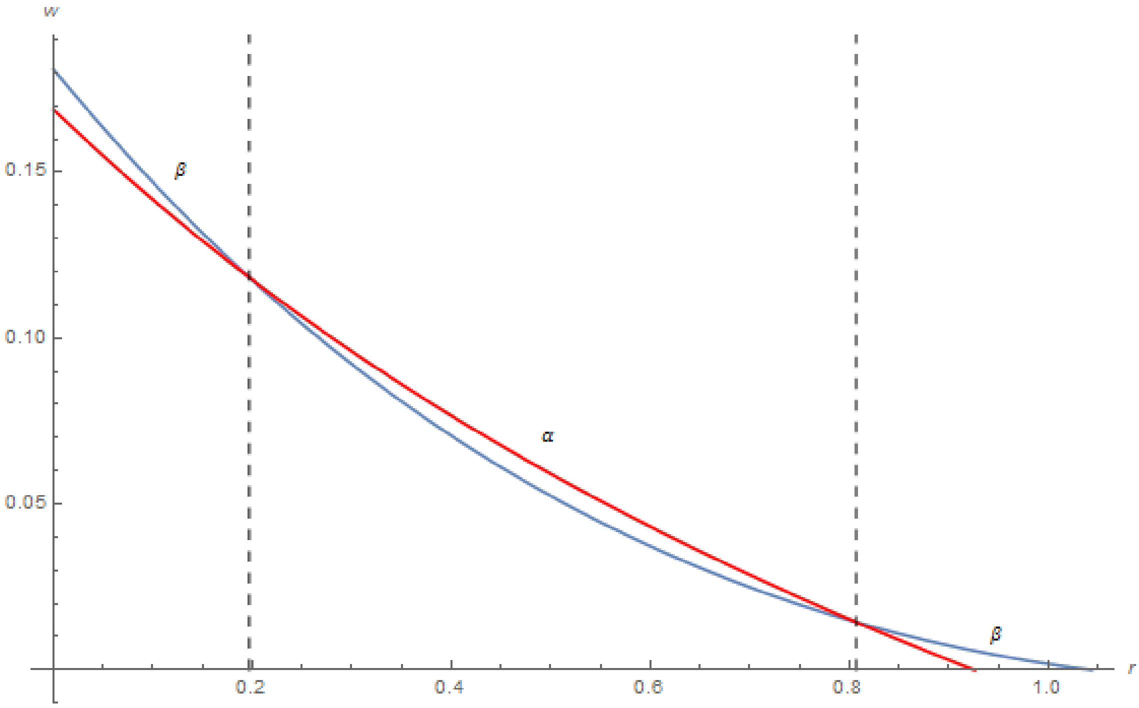

Figure 1: Factor price frontier of Example 1

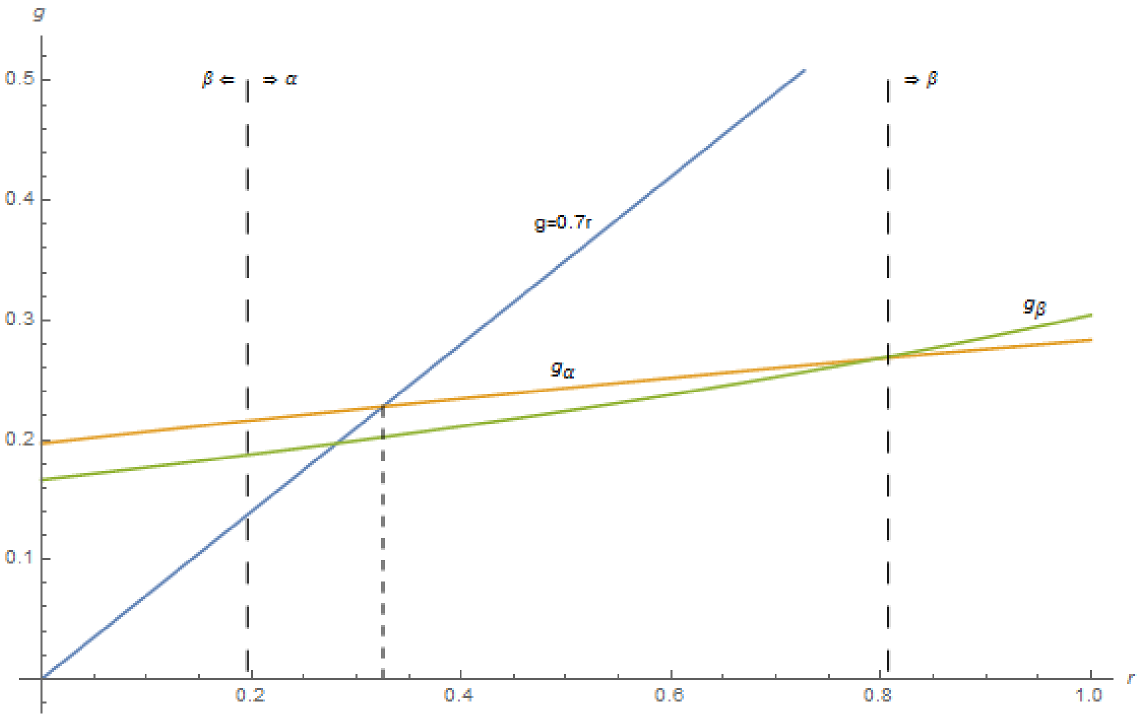

Figure 2: The rate of economic growth in (ILA, ILA) of Example 1

Note: Here, $g_\iota(r) = \rho_w \frac{\boldsymbol{p}^*(\boldsymbol{I}-\boldsymbol{A}^*)\boldsymbol{x}^*}{\boldsymbol{p}^*\boldsymbol{A}^*\boldsymbol{x}^*}$ for $\iota = \{\alpha, \beta\}$.

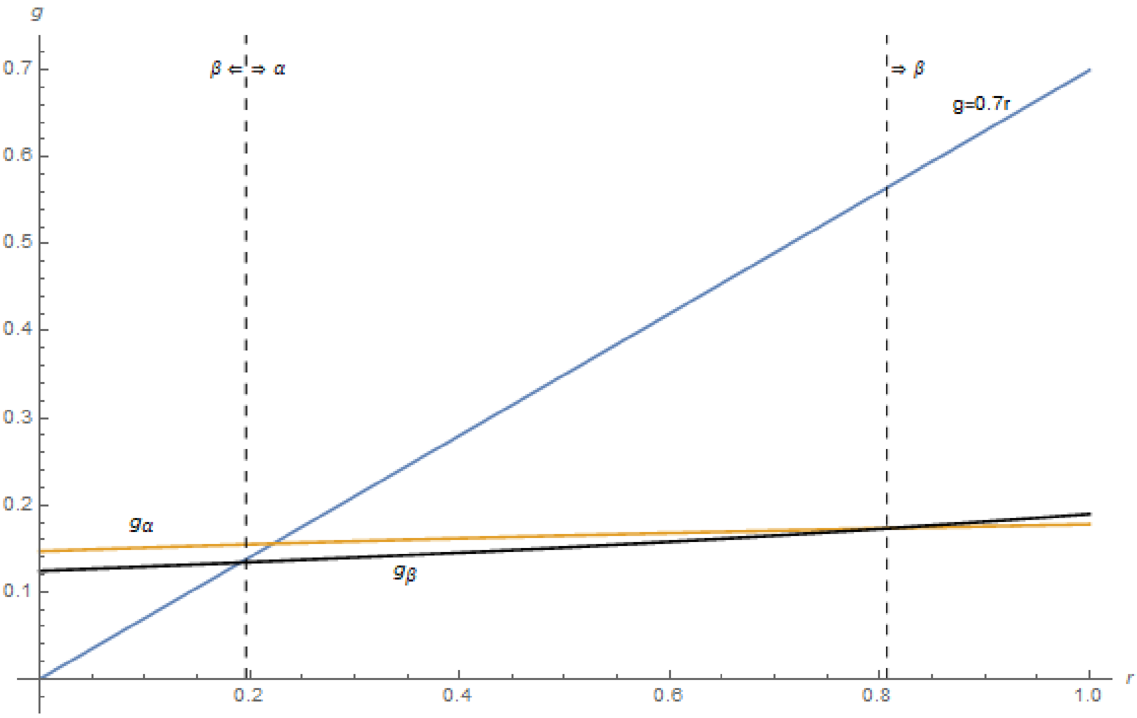

Figure 3: The rate of economic growth in (ILA, OLG) of Example 1

Note: Here, $g_\iota(r) = \frac{\rho_w}{\boldsymbol{p}^*\boldsymbol{A}^*\boldsymbol{x}^*}$ for $\iota = \{\alpha, \beta\}$.

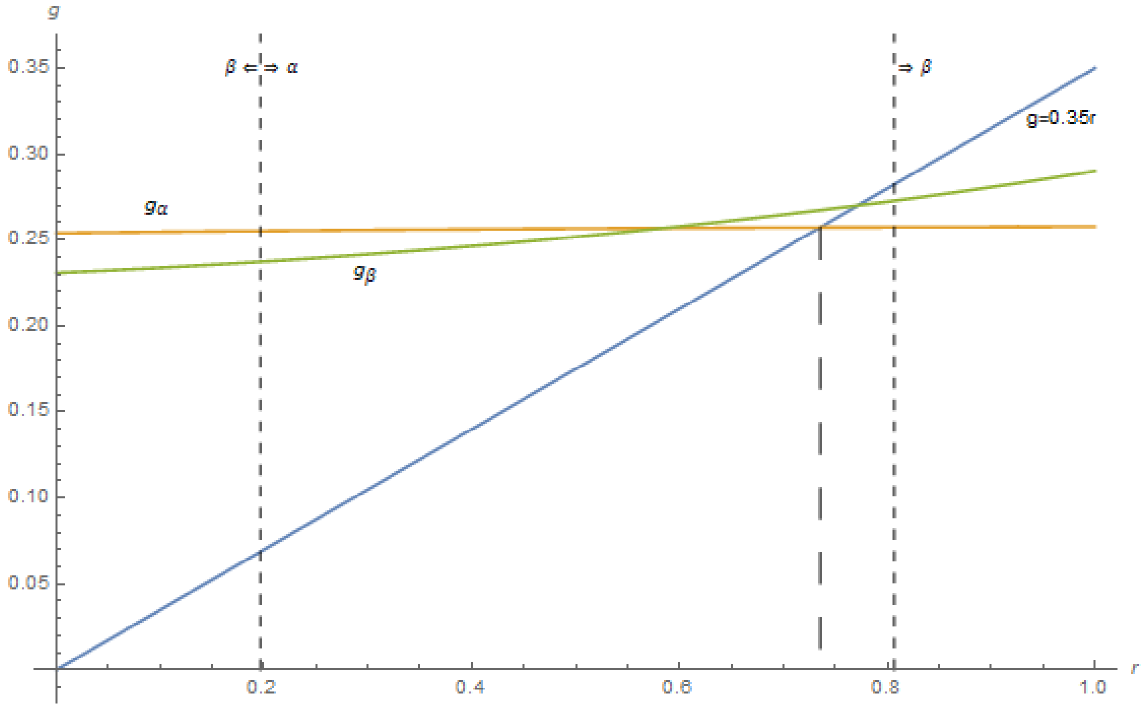

Figure 4: The case of $\rho_w > \rho_c$ ($\rho_c = 0.35, \rho_w = 0.6$) in Example 1

Note: Here, $g_\iota(r) = \frac{\rho_w}{\boldsymbol{p}^* \boldsymbol{A}^* \boldsymbol{x}^*}$ for $\iota = \{\alpha, \beta\}$.

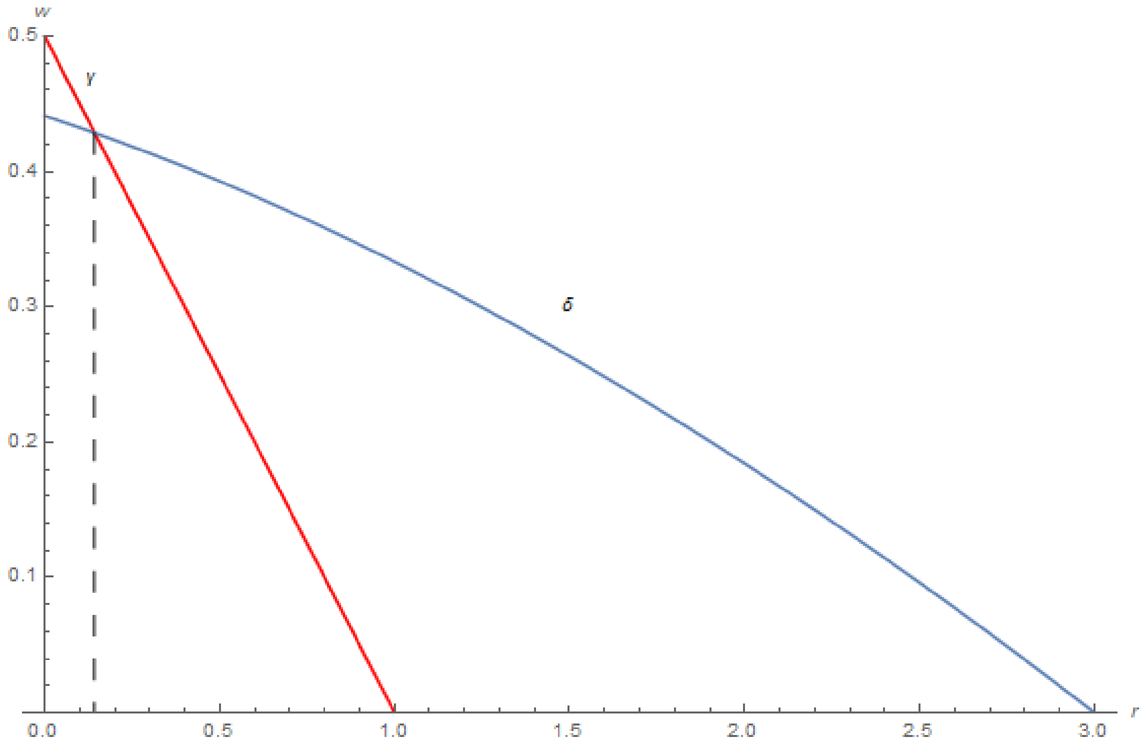

Figure 5: Factor price frontier of Example 2

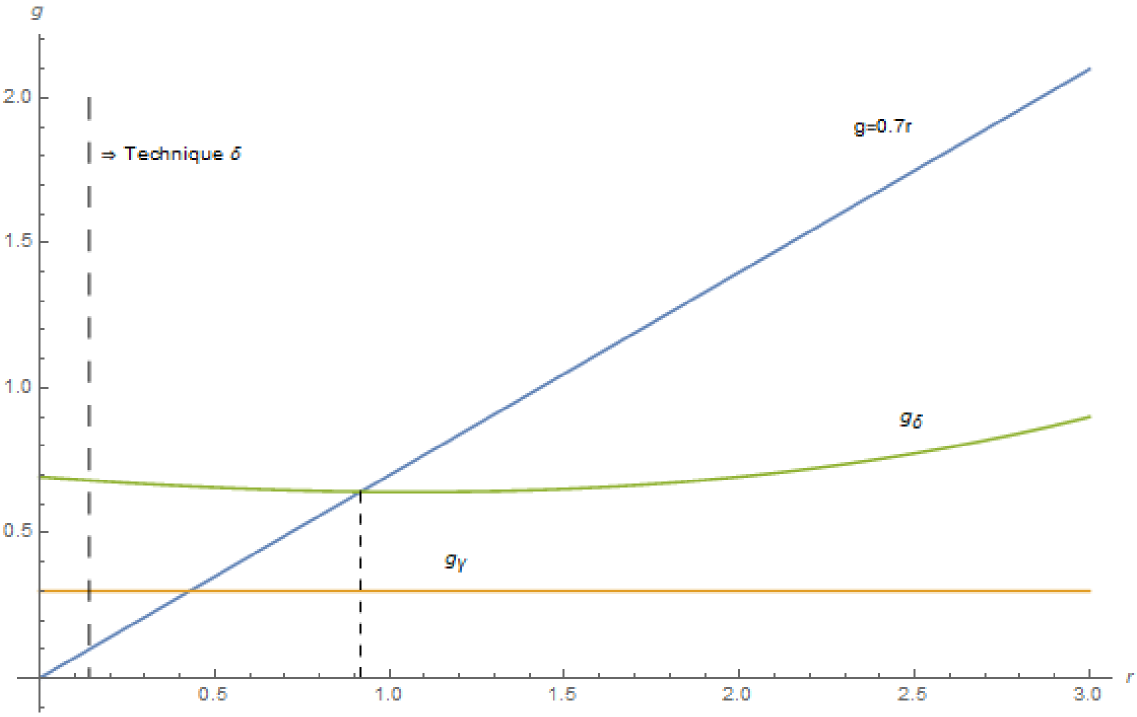

Figure 6: The rate of economic growth in (ILA, ILA) of Example 2

Note: Here, $g_\iota(r) = \rho_w \frac{\boldsymbol{p}^*(\boldsymbol{I}-\boldsymbol{A}^*)\boldsymbol{x}^*}{\boldsymbol{p}^*\boldsymbol{A}^*\boldsymbol{x}^*}$ for $\iota = \{\gamma, \delta\}$.

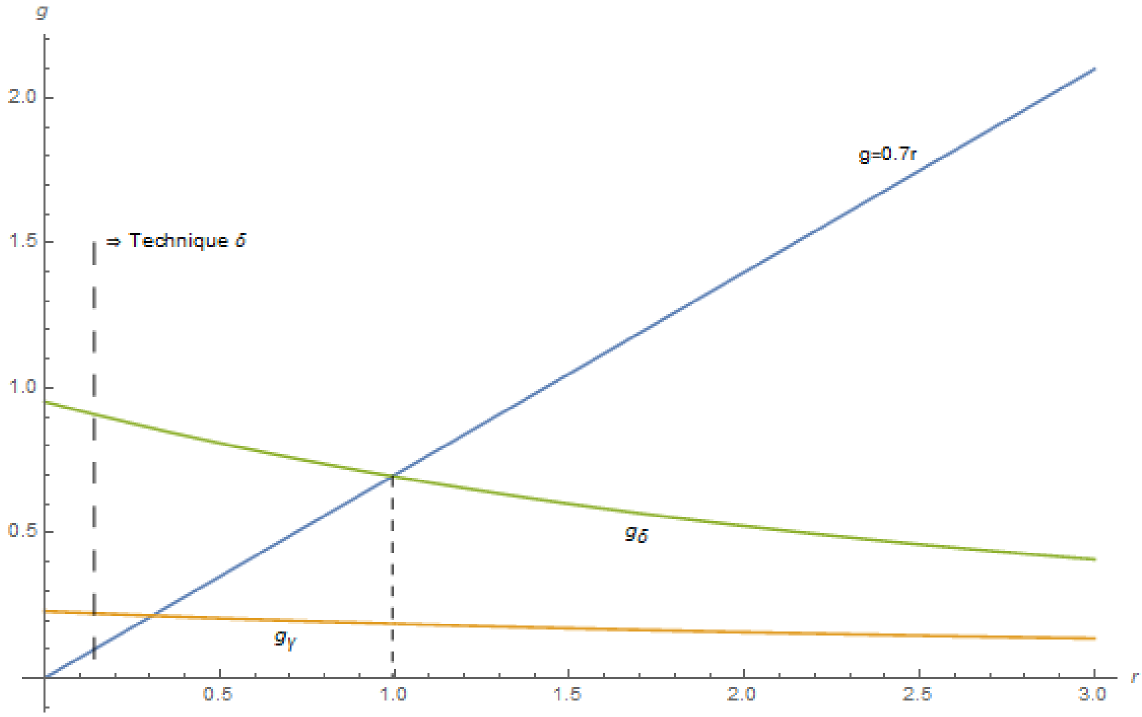

Figure 7: The rate of economic growth in (ILA, OLG) of Example 2

Note: Here, $g_\iota(r) = \frac{\rho_w}{\boldsymbol{p}^*\boldsymbol{A}^*\boldsymbol{x}^*}$ for $\iota = \{\gamma, \delta\}$.