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# Cardinal likelihoods: A joint characterization of the logarithmic and linear likelihood functions

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# Cardinal likelihoods: A joint derivation of the logarithmic and linear likelihood functions

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**Abstract**: A likelihood function is a real-valued function on the set of events of the sample space representing the likelihood of the occurrence of the events, and the logarithmic and linear (positive affine) likelihood functions are given by the logarithmic and linear transformations of a probability measure on the events, respectively. Specifying statistician's subjective likelihood of the events by a difference comparison relation, this paper provides some axioms for the relations to be represented cardinally by the two likelihood functions, the probability measures of which coincide with the unique subjective (conditional) probability measure determined by the relation. This result turns out that the difference of the axiomatizations for the likelihood functions is only the difference of the Lucian independence axioms, (i.e., difference in the definition of irrelevant events for the relations).

**Key words**: logarithmic likelihood function, linear likelihood function, qualitative conditional probability, difference comparison relation, cross-modality ordering, independence of irrelevant events.

## 1. Introduction

A likelihood function is a real-valued function defined on the events of a sample space representing the statistician's subjective likelihood of the occurrence of the events. Practically, the logarithmic and linear (positive affine) likelihood functions, which are given by the logarithmic and linear transformations of a probability measure on a sample space, respectively, are widely used in statistics. This paper attempts to derive jointly the two likelihood functions from the qualitative axioms stated in terms of a relation on the events, which specifies the statistician's subjective likelihood of the occurrence of the events.

The axiomatic foundation of the likelihood functions as the ordinal (order-preserving) indicators representing the relation has been established, because the likelihood function is given by the monotone transformation of the (subjective) probability measure of which axiomatic foundation is well-established.[1] The ordinal likelihood function is implicitly assumed when one computes the maximal likelihood estimators assuming the logarithmic likelihood function, because the logarithmic function is a monotone transformation and the maximal likelihood estimators are invalid against such a transformation.[2]

In some hypotheses testings, as shown in the Neyman and Pearson lemma, the likelihood ratio index is crucial for the optimal testing procedures, because the decisions of the testings are altered if we adapt another index such as the likelihood difference index.[3] Because the likelihood ratio and likelihood difference indices are derived from the logarithmic and linear likelihood functions, respectively, the functional forms of the likelihood function are meaningful and the likelihood functions are assumed as the cardinal functions in the hypotheses testings.[4] However the axiomatic foundation of the cardinal likelihood functions is not established. Namely the question what is the qualitative principles determining the logarithmic likelihood function is an

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1 See Fishburn (1986, 1994) for the surveys of the subjective probability theory.

2 See Myung (2003) for the maximum likelihood estimation.

3 See DeGroot (1970, Ch.8, Theorem 1, p.146).

4 Almost the same question is considered by Sober (2008, Ch 1, p.15 – p.17) in the literature of philosophy of science, where the cardinality of the confirmation measures closely related to the likelihood ratio index is discussed.

open question. Moreover, among the qualitative principles, the question what principles do explain the differences of the two cardinal likelihood functions is also open.

In this paper, we assume that the statistician's subjective likelihood of the occurrence of events in the sample space is specified by a difference comparison relation (called the relative likelihood relation) on the sample space, and we derive the two cardinal likelihood functions from some axioms on the relation as in the standard difference measurement theory,[5] without introducing the concepts in the mathematical physics and information theory such as the entropy and the complexity measure. Concretely, we provide the axioms on the relation not only for the existence of a qualitative (subjective) probability measure satisfying the law of conditional probability, but also for the representability by the logarithmic likelihood function as a cardinal indicator representing the relation (i.e., the likelihood function is determined unique up to the linear transformations). Moreover, replacing one of the axioms with a new axiom, this paper derives the linear likelihood function as a cardinal indicator representing the relation. This joint derivation result clarifies the qualitative principles underlying the two likelihood functions and points out the principles explaining the difference of the two likelihood functions.

In order to introduce some axioms for the qualitative conditional probability, we extend the sample space by adjoining the sample space to an auxiliary experiment of which events are Borel subsets in the unit interval [0, 1],[6] and then we introduce some axioms which specifies the

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[5] See Krantz *et al*. (1971, Ch. 4) and Luce and Suppes (2002, Difference Measurement, p.16) for the measurement theory.

[6] We can assume that the auxiliary experiment is conducted repeatedly many times and the subjective probability on [0, 1] can be interpreted as the objective probability which determined by the (ideal) limit of the frequencies of the realizations of the repeated experiments. While the original experiment for the original sample space can be interpreted as the one-shot experiment. Hence the difference relation includes the comparisons between the pairs of the events in the two distinct sample spaces can be recognized as a cross-modality ordering. For the cross-modality ordering, see Krantz et al. (1971, Ch. 4, Section 4.6), Roberts (1979, Ch.4, Section 4.4), Luce (2012) and Nakamura (2015). In particular, the subjective probability and the state-dependent utility are jointly derived by Nakamura (2015) from the axioms on the cross-modality orderings.

conditions of the relation using the Euclidean topology and linear operations in [0, 1]. The extended sample space is used by De Finetti (1970, Section 6), DeGroot (1970) and French (1982) to derive a qualitative (un-conditional) probability measure,[7] introducing the monotone continuity axiom, which specifies a topological property of the relation. In this paper, we introduce not only the continuity axiom, but also the independence of unit axiom as in Luce (1959, Ch.1, Section F) to specify an algebraic property of the relation.

Practically, in the next section, the basic sample space is introduced as an abstract measurable space, and we adjoin the sample space to the auxiliary experiment. In the section 3, we show that the five axioms in French (1982), including the continuity axiom, are necessary and sufficient for the existence of a qualitative (un-conditional) probability on the extended sample space (Theorem 1).[8] In the section 4, we prove that the five axioms for Theorem 1 and the three axioms added here, including the independence of unit axiom, are necessary and sufficient for the existence of the qualitative conditional probability (Theorem 2).

As the main result of this paper, in the section 5, we introduce three additional axioms including a Lucian independence axiom, and we show that the eleven axioms are the necessary and sufficient for the likelihood relation to be represented by the logarithmic likelihood function (Theorem 3), and the linear likelihood function is axiomatically derived (Theorem 4) only by replacing the Lucian independence axiom in Theorem 3 with another Lucian independence axiom.[9]

## 2. The likelihood relations and the likelihood functions

This section introduces the basic sample space as an abstract measurable space $(S, \mathcal{B}_S)$, and then we adjoin the sample space to an auxiliary experiment of which events are Borel sets in the unit interval $T \equiv [0, 1]$. Moreover, the relative likelihood relation is defined by a binary relation on the

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[7] See Fishburn (1986, Section 6) for the survey of the subjective probability theory based on the auxiliary experiment.

[8] Although French (1982, Section 3, Axiom SP5) assumes the Herstein and Milnor's (1953) continuity axiom also, we prove Theorem 1 without assuming the Herstein and Milnor's continuity axiom. Namely, our continuity axiom implies Herstein and Milnor's continuity axiom under the other four axioms.

[9] The two independence axioms are variants of Luce's (1959) independence axiom.

pairs of the conditional events of the extended sample space, and we define the logarithmic and linear likelihood function formally.

Let $S$ be the set of possible futures (a sample space). We assume that $S \cap [0, 1] = \phi$, where $\phi$ is the empty set. Let $\mathcal{B}_S$ be a σ-field of subsets of $S$, (i.e., $\mathcal{B}_S$ is a set of subsets of $S$ which is closed under complementation and σ-additivity). A set in $\mathcal{B}_S$ is called an event in $S$. Specifically, $S$ is called the *total* event of $\mathcal{B}_S$. The set of null events in $\mathcal{B}_S$ is denoted by $\mathcal{N}_S$. We assume that $S \notin \mathcal{N}_S$ and $\phi \in \mathcal{N}_S$. For a given pair of events $A$ and $B$ in $\mathcal{B}_S$ such that $B \notin \mathcal{N}_S$, a *conditional event* is denoted by an ordered pair $A|B$, where $A|B$ means an event $A$ conditioned on an event $B$. The set of all conditional events in $S$ is defined by $\Gamma_S = \{ A|B : A \in \mathcal{B}_S, B \in \mathcal{B}_S \text{ and } B \notin \mathcal{N}_S \}$.

Let $\mathcal{I}_T$ be the set of all intervals in $T \equiv [0, 1]$, and let $\mathcal{B}_T$ be the set of all Borel subsets in $T$, which is the minimal σ-field on $T$ containing $\mathcal{I}_T$.[10] A set in $\mathcal{B}_T$ is called an event in $T$. Specifically, $T$ is called the *total* event of $\mathcal{B}_T$. The set of null events in $\mathcal{B}_T$ is denoted by $\mathcal{N}_T$. We assume that $T \notin \mathcal{N}_T$ and $\phi \in \mathcal{N}_T$. A *conditional event* in $T$ is denoted by an ordered pair $A|B$ for $A, B \in \mathcal{B}_T$ such that $B \notin \mathcal{N}_T$, and the set of all conditional events in $T$ is defined by $\Gamma_T = \{ A|B : A \in \mathcal{B}_T, B \in \mathcal{B}_T \text{ and } B \notin \mathcal{N}_T \}$.

A *difference* on $\Gamma_S$ is a transition (path) from a conditional event $A|B \in \Gamma_S$ to a conditional event $C|D \in \Gamma_S$, and a *difference* on $\Gamma_T$ is a transition (path) from a conditional event $A|B \in \Gamma_T$ to a conditional event $C|D \in \Gamma_T$. In both cases, the difference from $A|B$ to $C|D$ is denoted by an ordered pair $(A|B, C|D)$. The sets of all admissible differences on $\Gamma_S$ and $\Gamma_T$ are defined by

$\Theta_S = \{ (A|B, C|D) : A|B \in \Gamma_S, C|D \in \Gamma_S \text{ and } A \cap B \notin \mathcal{N}_S \}$ and

$\Theta_T = \{ (A|B, C|D) : A|B \in \Gamma_T, C|D \in \Gamma_T \text{ and } A \cap B \notin \mathcal{N}_T \}$, respectively.

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[10] A singleton in $\mathcal{B}_T$ is denoted by $\{ t \}$ or $[t, t]$ for all $t \in T$, and we assume that $[t, t] \in \mathcal{I}_T$ for all $t \in T$. It holds by $S \cap [0, 1] = \phi$ that $\mathcal{B}_S \cap \mathcal{B}_T = \{ \phi \}$. We define the set-difference operation $A - B$ by $A - B \equiv \{x \in A : x \notin B\}$ for any $A, B \in \mathcal{B}_S \cup \mathcal{B}_T$ (i.e., $A - B$ is the relative complement of $B$ in $A$.) For example, $S - S = T - T = \phi$, $S - \phi = S$ and $T - \phi = T$.

A *relative likelihood relation* $\precsim$ on $\Theta_S \cup \Theta_T$ is a complete and transitive binary relation on $\Theta_S \cup \Theta_T$. The expression $(A|B, C|D) \precsim (A^*|B^*, C^*|D^*)$ means that the transition from $A|B$ to $C|D$ gives more added likelihood than the transition from $A^*|B^*$ to $C^*|D^*$. The symmetric and asymmetric parts of $\precsim$ are denoted by $\sim$ and $\succ$, respectively.

A function $F: \Gamma_S \cup \Gamma_T \to \mathbb{R} \cup \{-\infty\}$ is called a *likelihood function representing a relative likelihood relation* $\precsim$ if and only if

$$(A|B, C|D) \precsim (A^*|B^*, C^*|D^*) \Leftrightarrow F(C|D) - F(A|B) \geq F(C^*|D^*) - F(A^*|B^*) \qquad (1)$$

for all $(A|B, C|D), (A^*|B^*, C^*|D^*) \in \Theta_S \cup \Theta_T$. For the arithmetic rules for the extended real number $-\infty$, we assume that

$$-\infty = -\infty + x \text{ and } -\infty < x \text{ for all real numbers } x \in \mathbb{R}. \qquad (2)$$

In order to define the logarithmic and linear likelihood functions, we need a definition: a real-valued function $\pi(\cdot)$ on $\mathcal{B}_S \cup \mathcal{B}_T$ is called a probability function if and only if the following three conditions hold:

(**i**) The restriction of $\pi$ on $\mathcal{B}_S$ coincides with a probability measure on S.[11]

(**ii**) The restriction of $\pi$ on $\mathcal{B}_T$ coincides with a probability measure on T.

(**iii**) $\pi(A) > 0 \Leftrightarrow A \notin \mathcal{N}_S \cup \mathcal{N}_T$ for all $A \in \mathcal{B}_S \cup \mathcal{B}_T$.

For a given probability function $\pi$, a likelihood function $F^1(A|B)$ is defined to be *logarithmic* with respect to a probability function $\pi$ if and only if

$$F^1(A|B) = \log[\pi(A \cap B)/\pi(B)] \text{ for all } A|B \in \Gamma_S \cup \Gamma_T, \qquad (3)$$

and a likelihood function $F^2(A|B)$ is defined to be *linear* with respect to a probability function $\pi$ if and only if there exists $a > 0$ and $b$ such that

$$F^2(A|B) = a \cdot [\pi(A \cap B)/\pi(B)] + b \text{ for all } A|B \in \Gamma_S \cup \Gamma_T. \qquad (4)$$

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[11] A probability measure on S is a real-valued function $p$ on $\mathcal{B}_S$ such that: (i) $0 \leq p(A) \leq 1$ for $A \in \mathcal{B}_S$; (ii) $p(\phi)=0$, $p(S)=1$; (iii) $p$ is countably additive on $\mathcal{B}_S$. For the countable additivity, see Rosenthal (2006, Section 2.1, page 7) and Billingsley (1995, Ch. 1, Section 2, page 17).

## 3. The derivation of the qualitative (subjective) probability measure

This section provides the axioms on the relative likelihood relation for ensuring the existence of a subjective (un-conditional) probability measure representing a subrelation induced by the relation on the un-conditional events, $\mathcal{B}_S \cup \mathcal{B}_T$, based on DeGroot's (1970, Section 6.2) and French's (1982) results. First we define the subrelation: for a given relative likelihood relation $\precsim$ on $\Theta_S \cup \Theta_T$, a subrelation $\precsim'$ on $\mathcal{B}_S \cup \mathcal{B}_T$ is defined by

$$A \precsim' B \iff (\tau(A) \mid \tau(A), A \mid \tau(A)) \precsim (\tau(B) \mid \tau(B), B \mid \tau(A)) \text{ for all } A, B \in \mathcal{B}_S \cup \mathcal{B}_T. \qquad (5)$$

where $\tau(\cdot)$ is a set-valued function on $\mathcal{B}_S \cup \mathcal{B}_T$ defined by $\tau(A) = S$ if $A \in \mathcal{B}_S$; $\tau(A) = T$ if $A \in \mathcal{B}_T$. The expression $A \precsim' B$ means that an event A is (at least) more likely to occur than an event B, and the binary relation $\precsim'$ is called the *direct level relation* of $\precsim$. The direct level relation $\precsim'$ is complete and transitive. The symmetric and asymmetric parts of $\precsim'$ are denoted by $\sim'$ and $\succ'$, respectively.

If an axiom is stated in terms of the direct level relation $\precsim'$ of $\precsim$, the axiom is directly translated into the axiom in terms of the original relation $\precsim$ by way of the equivalence of (5). Practically, the direct level relation $\precsim'$ on $\mathcal{B}_S \cup \mathcal{B}_T$ corresponds to the relation assumed in subjective probability theory by developed by DeGroot (1970) and French (1982) and then we can re-state some of French's axioms in our setting:[12]

$\mathbf{L_1}$ (**Total and null events**): (**i**) $S \sim' T$, (**ii**) $A \precsim' \phi$ for all $A \in \mathcal{B}_S \cup \mathcal{B}_T$, (**iii**) $A \in \mathcal{N}_S \cup \mathcal{N}_T \Leftrightarrow A \sim' \phi$ for all $A \in \mathcal{B}_S \cup \mathcal{B}_T$.

$\mathbf{L_2}$ (**Additivity**): For any $A, B, C, D \in \mathcal{B}_S \cup \mathcal{B}_T$ such that $\tau(A) = \tau(B)$, $\tau(C) = \tau(D)$, if $A \cap B = \phi$ and $C \cap D = \phi$, then it holds that: (**i**) $(A \precsim' C, B \precsim' D) \Rightarrow A \cup B \precsim' C \cup D$; (**ii**) $(A \succ' C, B \precsim' D) \Rightarrow A \cup B \succ' C \cup D$.

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[12] Some of the axioms are introduced by DeGroot (1970). Concretely, the axiom $L_2$ is introduced by DeGroot (1970, Section 6.2, Assumption $SP_2$), and the axiom $L_3$ is introduced by DeGroot (1970, Section 6.2, Assumption $SP_4$). The axioms $L_4$ and $L_5$ are closely related to DeGroot (1970, Section 6.2, Assumption $SP_5$).

$\mathbf{L_3}$ (**Monotone continuity**): Let $\{ B_n \}$ be a sequence of events in $\mathcal{B}_S$ or $\mathcal{B}_T$. If $B_n \supset B_{n+1}$ for all $n$, and if there exists $A \in \mathcal{B}_S \cup \mathcal{B}_T$ such that $B_n \succsim' A$ for all $n$, then $\cap B_n \succsim' A$.

$\mathbf{L_4}$ (**Positivity**): $\mathbf{sup}\, A > \mathbf{inf}\, A \Leftrightarrow A \succ' \phi$ for all $A \in \mathcal{I}_T$.

$\mathbf{L_5}$ (**Invariance against parallel shifts to the right**): $A \sim' (A + c)$ for all $A \in \mathcal{I}_T$ and all $c \in [0, 1 - \sup A]$, where $A + c \equiv \{ x \in T : x = y + c \text{ for some } y \in A \}$.

The axiom $\mathbf{L_1}$ characterizes simply the total and null events. The axiom $\mathbf{L_2}$ used for the resulting probability measure satisfies the finite additivity and the axiom $\mathbf{L_3}$ used for the σ-additivity. The axioms $\mathbf{L_4}$ and $\mathbf{L_5}$ are standard axioms characterizing the Lebesgue measure $\mu$ on $\mathcal{B}_T$, which are stated using the geometric or algebraic properties of $[0, 1]$. Then we have the following theorem:

**Theorem 1** (French 1982, Section 3, Theorem): (**i**) If a relative likelihood relation $\succsim$ on $\Theta_S \cup \Theta_T$ satisfies the axioms $\mathbf{L_1} - \mathbf{L_5}$, then for each $A \in \mathcal{B}_S \cup \mathcal{B}_T$ there exists a real number $\pi_A \in [0, 1]$ uniquely such that $A \sim' [0, \pi_A]$, where $\succsim'$ is the direct level relation of $\succsim$. (**ii**) A relative likelihood relation $\succsim$ on $\Theta_S \cup \Theta_T$ satisfies the axioms $\mathbf{L_1} - \mathbf{L_5}$ if and only if there exists a (unique) probability function $\pi$ on $\mathcal{B}_S \cup \mathcal{B}_T$ such that $A \succsim' B \Leftrightarrow \pi(A) \geq \pi(B)$ for all $A, B \in \mathcal{B}_S \cup \mathcal{B}_T$, where $\succsim'$ is the direct level relation of $\succsim$, and that the restriction of $\pi$ on $\mathcal{B}_T$ coincides with the Lebesgue (probability) measure $\mu$ on $\mathcal{B}_T$. The probability function $\pi$ is given by Theorem 1(i) under $\mathbf{L_1} - \mathbf{L_5}$.[13]

## 4. The derivation of the qualitative conditional probability measure

For a given relative likelihood relation $\succsim$ on $\Theta_S \cup \Theta_T$, a subrelation $\succsim''$ on $\Gamma_S \cup \Gamma_T$ is defined by

$$A|B \succsim'' C|D \Leftrightarrow (\tau(B)|\tau(B), A|B) \succsim (\tau(D)|\tau(D), C|D) \text{ for all } A|B, C|D \in \Gamma_S \cup \Gamma_T. \quad (6)$$

The expression $A|B \succsim'' C|D$ means that a conditional event $A|B$ is (at least) more likely to

[13] We provide the proof of Theorem 1 in the section 6 in this paper for the completeness of the arguments. For the Lebesgue measure $\mu$ on $\mathcal{B}_T$, see Rosenthal (2006, Section 2.4, Theorem 2.4.4, page 16) and Billingsley (1995, Ch. 1, Section 2, Theorem 2.2, page 26).

occur than a conditional event $C|D$, and the binary relation $\precsim''$ is called the *conditional level relation* of $\precsim$. The relation $\precsim''$ is complete and transitive. The symmetric and asymmetric parts of $\precsim''$ are denoted by $\sim''$ and $\succ''$, respectively. It holds by (5) and (6) that

$$A \precsim' B \Leftrightarrow A|\tau(A) \precsim'' B|\tau(B) \text{ for all } A, B \in \mathcal{B}_S \cup \mathcal{B}_T, \tag{7}$$

If an axiom is stated in terms of the conditional level relation $\precsim''$ of $\precsim$, the axiom is directly translated into the axiom in terms of the original relation $\precsim$ by way of the equivalence of (6). Practically, the conditional level relation of $\precsim$ just corresponds to the conditional likelihood relation in Luce (1968), and we can provide some axioms in terms of the conditional level relation $\precsim''$ for ensuring the existence of a *qualitative conditional* probability function determined by $\precsim$, which is defined by a probability function $\pi$ on $\mathcal{B}_S \cup \mathcal{B}_T$ such that

$$A|B \precsim'' C|D \Leftrightarrow \pi(A \cap B)/\pi(B) \geq \pi(C \cap D)/\pi(D) \text{ for all } A|B, C|D \in \Gamma_S \cup \Gamma_T.^{14} \tag{8}$$

We introduce the following three additional axioms for Theorem 2, which are stated in terms of the conditional level relation $\precsim''$ of $\precsim$:

$\mathbf{L_6}$ (**Consistency I**): (**i**) For A, B, C $\in \mathcal{B}_S \cup \mathcal{B}_T$, if $(A \subset C \succ' \phi$, $B \subset C$ and $A|C \precsim'' B|C)$ or $(C \subset B \succ' \phi$, $C \subset A \succ' \phi$ and $C|B \precsim'' C|A)$, then $A \precsim' B$. (**ii**) For A, B, C $\in \mathcal{B}_S \cup \mathcal{B}_T$, if $A \subset B$ and $C \subset D$, and if $A \sim' C$ and $B \sim' D \succ' \phi$, then $A|B \sim'' C|D$.

$\mathbf{L_7}$ (**Independence of unit**): For all $A, B \in \mathcal{I}_T$ with $A \subset B$ and all $c \in (0, 1]$, if $A|B \in \Gamma_T$ and $c{\cdot}A|c{\cdot}B \in \Gamma_T$, then $A|B \sim'' c{\cdot}A|c{\cdot}B$, where $c{\cdot}A \equiv \{x \in T : x = c{\cdot}y \text{ for some } y \in A\}$.

$\mathbf{L_8}$ (**Essentiality**): $A|B \sim'' (A \cap B)|B$ for all $A|B \in \Gamma_S \cup \Gamma_T$.

The axiom $\mathbf{L_6}$ requires the consistency between the two relations $\precsim'$ and $\precsim''$. The linear operation on [0, 1] means the change of the unit of [0, 1], and the axiom $\mathbf{L_7}$ means that the level relation $\precsim'$ is independent of the unit of [0, 1]. The axiom $\mathbf{L_7}$ is closely related to Luce's (1959,

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[14] Setting $B = \tau(A)$ and $D = \tau(C)$ in (8) above, we have by (7) that $A \precsim' C \Leftrightarrow A|\tau(A) \precsim'' C|\tau(C) \Leftrightarrow \pi(A) \geq \pi(C)$ for all $A, C \in \mathcal{B}_S \cup \mathcal{B}_T$, which implies that the restrictions of $\pi$ on $\mathcal{B}_S$ or $\mathcal{B}_T$ are qualitative (subjective) probability measures representing $\precsim'$.

Ch.1, Section F, p.28) independence of unit axiom, which is stated in terms of a numerical function in a different setting. The axiom $\mathbf{L_8}$ is standard and it is introduced by Luce (1968, Section 2, Axiom 4).[15] The main result of this section is the following theorem:

**Theorem 2**: A relative likelihood relation $\precsim$ on $\Theta_S \cup \Theta_T$ satisfies all through the axioms $\mathbf{L_1} - \mathbf{L_8}$ if and only if there exists a unique qualitative conditional probability function $\pi$ on $\mathcal{B}_S \cup \mathcal{B}_T$ determined by $\precsim$ and the restriction of $\pi$ on $\mathcal{B}_T$ coincides with the Lebesgue (probability) measure on $\mathcal{B}_T$. The probability function $\pi$ is given by Theorem 1(i) under $\mathbf{L_1} - \mathbf{L_8}$.

Setting $F^0(A \mid B) = \pi(A \cap B)/\pi(B)$ for all $A \mid B \in \Gamma_S \cup \Gamma_T$ in Theorem 2, it holds that

$$A \mid B \succsim' C \mid D \iff F^0(A \mid B) \geq F^0(C \mid D) \text{ for all } A \mid B, C \mid D \in \Gamma_S \cup \Gamma_T.$$

Hence it follows from Theorem 2 that there exists an ordinal likelihood function representing a conditional level relation of $\succsim$, if the relation $\succsim'$ satisfies all through the axioms $\mathbf{L_1} - \mathbf{L_8}$.

## 5. The joint derivation of the logarithmic and linear likelihood functions

For the next two theorems, we introduce some additional axioms.

$\mathbf{L_9}$ (**Consistency II**): If $(A \mid B, C \mid D), (A^* \mid B^*, C^* \mid D^*) \in \Theta_S \cup \Theta_T$, and if $A \mid B \sim'' A^* \mid B^*$ and $C \mid D \sim'' C^* \mid D^*$, then $(A \mid B, C \mid D) \sim (A^* \mid B^*, C^* \mid D^*)$.

$\mathbf{L_{10}}$ (**Inversion**): $(A \mid T, B \mid T) \precsim (C \mid T, D \mid T) \Rightarrow (D \mid T, C \mid T) \precsim (B \mid T, A \mid T)$ for all $(A \mid T, B \mid T), (C \mid T, D \mid T) \in \Theta_S \cup \Theta_T$ with $(B \mid T, A \mid T), (D \mid T, C \mid T) \in \Theta_S \cup \Theta_T$.

$\mathbf{L_{11}}$ (**Independence of irrelevant events I**): For all $A, B, C \in \mathcal{I}_T$ with $A \subset B$ and $A \succ' \phi$, if $B \cap C = \phi$, then $(A \mid T, B \mid T) \sim (A \mid (B \cup C), B \mid (B \cup C))$.

$\mathbf{L^*_{11}}$ (**Independence of irrelevant events II**): For all $A, B, C \in \mathcal{I}_T$ with $A \subset B$ and $A \succ' \phi$, if $B \cap C = \phi$, then $(A \mid T, B \mid T) \sim ((A \cup C) \mid T, (B \cup C) \mid T)$.

The axiom $\mathbf{L_9}$ requires the consistency between the two relations $\succsim''$ and $\succsim$. The axiom $\mathbf{L_{10}}$ is

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[15] Luce (1968) provides the axioms which are only sufficient for the existence of a subjedtive conditional probability. See Fishburn (1986, Section 7) for the survey of the subjective conditional probability theory.

a standard condition as in Krantz *et al*. (1971, Ch. 4, Section 4.4, Definition 2, Axiom 2). The two axioms $\mathbf{L_{11}}$ and $\mathbf{L^*_{11}}$ can be recognized as the two variants of Luce's (1959, Section 1.C., Lemma 2) independence axiom (independence of irrelevant alternatives) stated in terms of a relative likelihood relation, although Luce's original independence axiom is stated in terms of the choice probability in a different setting.[16] The irrelevant event is specified by the conditioning event in the axiom $\mathbf{L_{11}}$, and the irrelevant event is specified by the conditioned event in the axiom $\mathbf{L^*_{11}}$. Namely, both of the axioms specify the qualitative conditions, using neither topological nor algebraic (linear) properties of the relation, except for the Boolean operstions. As the main result of this paper, we have the following theorems:

**Theorem 3**: (**i**) A relative likelihood relation $\precsim$ on $\Theta_S \cup \Theta_T$ satisfies all through the axioms $\mathbf{L_1}$ – $\mathbf{L_{10}}$ and $\mathbf{L_{11}}$ if and only if the relation $\precsim$ is represented by a logarithmic likelihood function, the probability function of which coincides with the unique subjective probability function determined by $\precsim$. (**ii**) Suppose that a relative likelihood relation $\precsim$ satisfies all the axioms in the assertion (**i**) above, and let $F^1$ be the logarithmic likelihood function. A real-valued function $F$ on $\Gamma_S \cup \Gamma_T$ is a likelihood function representing the relation $\precsim$ if and only if there exists $a > 0$ and $b$ such that $F(A \mid B) = a \cdot F^1(A \mid B) + b$ for all $A \mid B \in \Gamma_S \cup \Gamma_T$. Moreover, for any two likelihood functions $F$ and $F^*$ representing the relation $\precsim$, there exists $a > 0$ and $b$ such that $F(A \mid B) = a \cdot F^*(A \mid B) + b$ for all $A \mid B \in \Gamma_S \cup \Gamma_T$.

**Theorem 4**: (**i**) A relative likelihood relation $\precsim$ on $\Theta_S \cup \Theta_T$ satisfies all through the axioms $\mathbf{L_1}$ – $\mathbf{L_{10}}$ and $\mathbf{L^*_{11}}$ if and only if the relation $\precsim$ is represented by a linear likelihood function, the probability function of which coincides with the unique subjective probability function deternined by $\precsim$. (**ii**) Suppose that a relative likelihood relation $\precsim$ satisfies all the axioms in the assertion (**i**) above, and let $F^2$ be the linear likelihood function. A real-valued function $F$ on $\Gamma_S \cup \Gamma_T$ is a likelihood function representing the relation $\precsim$ if and only if there exists $a > 0$ and $b$ such that

[16] For the choice theoretic interpretation of Luce's independence axiom, see Ray (1973) and Echenique, *et al*. (2018) and the references.

$F(A|B) = a \cdot F^2(A|B) + b$ for all $A|B \in \Gamma_S \cup \Gamma_T$. Moreover, for any two likelihood functions $F$ and $F^*$ representing the relation $\gtrsim$, there exists $a > 0$ and $b$ such that $F(A|B) = a \cdot F^*(A|B) + b$ for all $A|B \in \Gamma_S \cup \Gamma_T$.

This joint derivation result implies that the two independence axioms are independent in the axiomatizations. To prove that the axiom $\mathbf{L_{11}}$ is independent of the other axioms in Theorem 3, it suffices to prove that the relation induced by a linear likelihood function does not satisfies the axiom $\mathbf{L_{11}}$, because the induced relation satisfies the other axioms as shown by Theorem 4.

Specifically, for a given domain $\Gamma_S \cup \Gamma_T$, let $F^2(A|B) = \pi(A \cap B)/\pi(B)$ be a linear likelihood function on $\Gamma_S \cup \Gamma_T$,, where $\pi$ is a probability function on $\mathcal{B}_S \cup \mathcal{B}_T$. Setting A = [0, 1/2], B = [0, 3/4] and C = (3/4, 1], it holds that $F^2(B|T) - F^2(A|T) = (3/4) - (1/2) = 1/4$ and $F^2(B|(B \cup C)) - F^2(A|(B \cup C)) = (3/4)/(3/4+1/4) - (1/4)/(3/4+1/4) = 1/2$, which implies that $(A|(B \cup C), B|(B \cup C)) \succ^2 (A|T, B|T)$, where $\succ^2$ is induced from $F^2$. Hence the relation induced by $F^2$ does not satisfies the axiom $\mathbf{L_{11}}$ .

By amost the same manner, we can prove that the axiom $\mathbf{L^*_{11}}$ is independent of the other axioms in Theorem 4. Let $F^1(A|B) = \log[\, \pi(A \cap B)/\pi(B)\, ]$ be a logarithmic likelihood function. It holds that $F^1(B|T) - F^1(A|T) = \log(3/4) - \log(1/2) = \log(3/2)$ and $F^1((B \cup C)|T) - F^1((A \cup C)|T) = \log(3/4+1/4) - \log(1/4+1/4) = \log 2$, which implies that $((A \cup C)|T, (B \cup C)|T) \succ^1 (A|T, B|T)$, where $\succ^1$ is induced from $F^1$. Hence the relation determined by $F^1$ does not satisfies the axiom $\mathbf{L^*_{11}}$, and then the axiom $\mathbf{L^*_{11}}$ is independent of the other axioms in Theorem 4.

A relative likelihood relation is a subjective concept, because it can be recognized as a specific data derived in a hypothetical experiment, where the statistician's responses are noted as a Yes-No sequence for the sequence of questions such as "Do you feel that the transition from $A|B$ to $C|D$ gives more added likelihood than the transition from $A^*|B^*$ to $C^*|D^*$ ?". However, if a relative likelihood relation satisfies the axioms in the theorems, the joint derivation result above implies that the axioms determine the functional forms completely and there is no functional variety specific to the statistician.

## 6. The proof of Theorem 1 and Theorem 2

**Proof of Theorem 1** (**i**) : Suppose that a relative likelihood relation $\succsim$ on $\Theta_S \cup \Theta_T$ satisfies all the axioms $\mathbf{L_1} - \mathbf{L_5}$. We need a lemma, which is proved in Appendix:

**Lemma 1**: If $\succsim$ satisfies all through the axioms $\mathbf{L_1} - \mathbf{L_5}$, then the following nine assertions hold: (**i**) $\{ a \} \sim' \phi$ for all $a \in T$. (**ii**) $[a, b] \succ' \phi$ for all $a, b \in T$ with $a < b$. (**iii**) $[a, b] \sim' [a, b) \sim' (a, b] \sim' (a, b)$ for all $a, b \in T$ with $a < b$. (**iv**) $[0, a] \succsim' [0, b] \Leftrightarrow a \geq b$ for all $a, b \in T$. (**v**) $m(J) \geq m(K) \Leftrightarrow J \succsim' K$ for all $J, K \in \mathcal{I}_T$, where $m(J) \equiv \mathbf{sup}\, J - \mathbf{inf}\, J$ is the length of $J \in \mathcal{I}_T$. (**vi**) $A \succsim' B \Rightarrow \tau(B) - B \succsim' \tau(A) - A$ for all $A, B \in \mathcal{B}_S \cup \mathcal{B}_T$ with $\tau(A) = \tau(B)$, where $\tau(B) - B \equiv \{x \in \tau(B) : x \notin B\}$. (**vii**) Let $\{ B_n \}$ be a sequence of events in $\mathcal{B}_S \cup \mathcal{B}_T$ satisfying $\{ B_n \} \subset \mathcal{B}_S$ or $\{ B_n \} \subset \mathcal{B}_T$. If $B_n \subset B_{n+1}$ for all $n$ and if there exists $A \in \mathcal{B}_S \cup \mathcal{B}_T$ such that $A \succsim' B_n$ for all $n$, then $A \succsim' \cup B_n$. (**viii**) If a convergent sequence $\{ x_n \}$ in $T$ satisfies $x_n \geq x_{n+1}$ for all $n$ and if there exists $A \in \mathcal{B}_S \cup \mathcal{B}_T$ such that $[0, x_n] \succsim' A$ for all $n$, then $[0, \lim x_n] \succsim' A$. (**ix**) If a convergent sequence $\{ x_n \}$ in $T$ satisfies $x_n \leq x_{n+1}$ for all $n$ and if there exists $A \in \mathcal{B}_S \cup \mathcal{B}_T$ such that $A \succsim' [0, x_n]$ for all $n$, then $A \succsim' [0, \lim x_n]$. (**x**) The two sets, $\{ x \in T : A \succsim' [0, x] \}$ and $\{ x \in T : [0, x] \succsim' A \}$ are non-empty and closed in $T$ for all $A \in \mathcal{B}_S \cup \mathcal{B}_T$.

Fix any $A \in \mathcal{B}_S \cup \mathcal{B}_T$. It holds by the connectedness of $T$ and Lemma 1(x) that there exists a real number $x \in T$ such that $A \sim' [0, x]$. The uniqueness of $x \in T$ is ensured by Lemma 1(iv). □

**Proof of Theorem 1** (**ii**) : Let $\succsim$ be a relative likelihood relation on $\Theta_S \cup \Theta_T$, and suppose that there exists a unique real-valued function $\pi$ on $\mathcal{B}_S \cup \mathcal{B}_T$ satisfying the condition. Then we can prove easily that $\succsim$ satisfies all the axioms.

Conversely, suppose that a relative likelihood relation $\succsim$ satisfies all the axioms. Let $\pi$ be a real-valued function on $\mathcal{B}_S \cup \mathcal{B}_T$ defined by Theorem 1(i). The axiom $\mathbf{L_1}$(i) implies that $S \sim' T = [0, 1]$. Hence we have by Theorem 1(i) that $\pi(S) = \pi(T) = 1$. Moreover, it holds by $\mathbf{L_1}$(iii) and Lemma 1(iv) that $\pi(A) > 0 \Leftrightarrow A \notin \mathcal{N}_S \cup \mathcal{N}_T$ for all $A \in \mathcal{B}_S \cup \mathcal{B}_T$, which which implies $\pi(\phi) = 0$, because $\phi \in \mathcal{N}_S$.

We will prove that $\pi$ is finitely additive on $\mathcal{B}_S$. Fix any $A, B \in \mathcal{B}_S$ with $A \cap B = \phi$. It holds by Theorem 1(i) that $A \sim' [0, \pi(A)]$ and $B \sim' [0, \pi(B)]$. It follows from Lemma 1(iii, v) that $A \sim' [0, \pi(A)] \sim' [0, \pi(A))$ and $B \sim' [0, \pi(B)] \sim' [\pi(A), \pi(A)+\pi(B)]$. We have by $\mathbf{L}_2$ that $A \cup B \sim' [0, \pi(A)) \cup [\pi(A), \pi(A)+\pi(B)] = [0, \pi(A)+\pi(B)]$, which implies that $\pi(A \cup B) = \pi(A)+\pi(B)$. Hence $\pi$ is finitely additive on $\mathcal{B}_S$.

We will prove that $\pi$ is σ-additive on $\mathcal{B}_S$. It suffices to prove that if $\{ A_n \}$ is a sequence of events in $\mathcal{B}_S$ satisfing $A_{n+1} \subset A_n$ for all $n$ and if $\cap A_n = \phi$, then $\lim \pi(A_n) = 0$. Because $\pi$ is finitely additive on $\mathcal{B}_S$ and $A_n = A_{n+1} \cup (A_n - A_{n+1})$ for all $n$, we have that $\pi(A_n) \geq \pi(A_{n+1}) \geq 0$ for all $n$, which implies that $\{ \pi(A_n) \}$ is a bounded monotone sequence. Hence it holds by Klambauer (1986, Proposition 7.8, page 383) that $\lim \pi(A_n)$ exists and $\lim \pi(A_n) \geq 0$. Suppose that $\lim \pi(A_n) > 0$. Set $a = \lim \pi(A_n) > 0$. It holds by Theorem 1(i) that $A_n \sim' [0, \pi(A_n)]$. Because $\pi(A_n) \geq a > 0$ for all $n$, we have by Lemma 1(ii, v) that $A_n \sim' [0, \pi(A_n)] \succsim' [0, a] \succ' [0, 0] \sim' \phi$ for all $n$. It holds by $\mathbf{L}_3$ that $\cap A_n \succ' \phi$. This contradicts with $\cap A_n = \phi$. Hence $\lim \pi(A_n) = 0$ and we have that $\pi$ is σ-additive on $\mathcal{B}_S$.

We can prove that $\pi$ is σ-additive on $\mathcal{B}_T$ by almost the same manner in the proof of the σ-additivity of $\pi$ on $\mathcal{B}_S$ above. Moreover, we can prove that $\pi$ represents $\succsim'$. Practically, for all $A, B \in \mathcal{B}_S \cup \mathcal{B}_T$, it holds by Theorem 1(i) and Lemma 1(v) that $A \succsim' B \Leftrightarrow [0, \pi(A)] \succsim' [0, \pi(B)] \Leftrightarrow \pi(A) \geq \pi(B)$.

Finally, we will prove that the restriction of $\pi$ on $\mathcal{B}_T$ coincides with the Lebesgue measure $\mu$ on $\mathcal{B}_T$. For any intervals $J \in \mathcal{I}_T$, we have $J \sim' [0, m(J)]$ by Lemma 1(v), which implies $\pi(J) = m(J)$. Hence, we have by Carathéodory's extension theorem as in Rosenthal (2006, Proposition 2.5.8) that $\pi(A) = \mu(A)$ for all $A \in \mathcal{B}_T$. □

**Proof of Theorem 2**: Let $\succsim$ be a relative likelihood relation on $\Theta_S \cup \Theta_T$, and suppose that there exists a unique probability function $\pi$ on $\mathcal{B}_S \cup \mathcal{B}_I$ satisfying the condition in Theorem 2. Then we can prove easily that $\succsim$ satisfies all the axioms.

Conversely, suppose that a relative likelihood relation $\succsim$ satisfies all the axioms in Theorem

2. It folllows from Theorem 1(ii) that there exists a unique probability function $\pi$ on $\mathcal{B}_S \cup \mathcal{B}_T$ such that $A \precsim' B \Leftrightarrow [0, \pi(A)] \precsim' [0, \pi(B)] \Leftrightarrow \pi(A) \geq \pi(B)$. We will prove that $A \mid B \precsim'' C \mid D \Leftrightarrow (A \cap B) \mid B \precsim'' (C \cap D) \mid D \Leftrightarrow \pi(A \cap B)/\pi(B) \geq \pi(C \cap D)/\pi(D)$. We need a lemma:

**Lemma 2**: (**i**) Fix any $A_2 \mid A_1, A_4 \mid A_3, B_2 \mid B_1, B_4 \mid B_3 \in \Gamma_S \cup \Gamma_T$. If $A_i \sim' B_i$ for i = 1, 2, 3, 4, and if $A_{j+1} \subset A_j \succ' \phi$ and $B_{j+1} \subset B_j$ for j = 1, 3, then $A_2 \mid A_1 \precsim'' A_4 \mid A_3 \Leftrightarrow B_2 \mid B_1 \precsim'' B_4 \mid B_3$.

(**ii**) $[0, \beta] \mid [0, \alpha] \precsim'' [0, \delta] \mid [0, \gamma] \Leftrightarrow \beta/\alpha \geq \delta/\gamma$ for all $\alpha, \beta, \gamma, \delta \in T$ with $\alpha, \gamma > 0, \alpha \geq \beta, \gamma \geq \delta$.

(**iii**) $A \mid B \precsim'' C \mid D \Leftrightarrow \pi(A)/\pi(B) \geq \pi(C)/\pi(D)$ for any $A \mid B, C \mid D \in \Gamma_S \cup \Gamma_T$ with $A \subset B$ and $C \subset D$.

Fix any $A \mid B, C \mid D \in \Gamma_S \cup \Gamma_T$. It holds by the axiom $\mathbf{L_8}$ and Lemma 2(iii) that

$$A \mid B \precsim'' C \mid D \Leftrightarrow (A \cap B) \mid B \precsim'' (C \cap D) \mid D \Leftrightarrow \pi(A \cap B)/\pi(B) \geq \pi(C \cap D)/\pi(D). \qquad \square$$

## 7. The proof of Theorem 3 and Theorem 4

For the proof of the next two theorems, we need a lemma:

**Lemma 3**: If there is a likelihood function $F$ representing a relative likelihood relation $\precsim$, and if $F$ is logarithmic or linear with respect to a probability function $\pi$, then the probability function $\pi$ is a subjective probability function of $\precsim$.

Moreover, we need another subrelation of the relative likelihood relation $\precsim$: for a given relative likelihood relation $\precsim$ on $\Theta_S \cup \Theta_T$, a subrelation $\precsim_*$ on $\Delta_T$ is defined by

$$(A, B) \precsim_* (C, D) \Leftrightarrow (A \mid T, B \mid T) \precsim (C \mid T, D \mid T) \text{ for all } (A, B), (C, D) \in \Delta_T, \tag{9}$$

where $\Delta_T = \{ (A, B) \equiv (A \mid T, B \mid T) : A, B \in \mathcal{B}_T \text{ and } A \notin \mathcal{N}_T \}$. The relation $\precsim_*$ is complete and transitive. The symmetric and asymmetric parts of $\precsim_*$ are denoted by $\sim_*$ and $\succ_*$, respectively.

**Proof of Theorem 3 (i)**: Suppose that $\precsim$ is represented by a logarithmic likelihood function with respect to the probability function $\pi$ on $\mathcal{B}_S \cup \mathcal{B}_T$ satisfying the condition in Theorem 2. It holds by Lemma 3 and Theorem 2 that $\precsim$ satisfies $\mathbf{L_1} - \mathbf{L_8}$. Moreover, we can prove easily that $\precsim$ satisfies $\mathbf{L_9} - \mathbf{L_{11}}$.

Conversely, suppose that a relative likelihood relation $\precsim$ on $\Theta_S \cup \Theta_T$ satisfies all the axioms. We will show that $\precsim$ is represented by the likelihood function which is logarithmic with respect to the probability function $\pi$ on $\mathcal{B}_S \cup \mathcal{B}_T$ satisfying the condition in Theorem 2. We need the following two lemmas:

**Lemma 4**: Suppose that the relation $\precsim$ satisfies all the axioms $\mathbf{L_1} - \mathbf{L_{10}}$. (**i**) If $A \sim' A^* \succ' \phi$ and $B \sim' B^* \succ' \phi$ for $A, A^*, B, B^* \in \mathcal{B}_T$, and if $A \subset B$ and $A^* \subset B^*$, then $(A, B) \sim_* (A^*, B^*)$. (**ii**) $(A, B) \succ_* (C, D) \Leftrightarrow (D, C) \succ_* (B, A)$ for all $(A, B), (C, D) \in \Delta_T$ with $(B, A), (D, C) \in \Delta_T$.

**Lemma 5**: Suppose that the relation $\precsim$ satisfies all the axioms $\mathbf{L_1} - \mathbf{L_{11}}$. (**i**) $A|B \precsim'' C|D \Leftrightarrow (B, A) \precsim_* (D, C)$ for any $A|B, C|D \in \Gamma_T$ with $A \subset B$ and $C \subset D$. (**ii**) For all $A, B \in \mathcal{I}_T$ with $B \subset A \succ' \phi$ and all $c \in (0, 1]$, if $c{\cdot}A \succ' \phi$, then $(A, B) \sim_* (c{\cdot}A, c{\cdot}B)$, where $c{\cdot}A \equiv \{ x \in T : x = c{\cdot}y$ for some $y \in A \}$. (**iii**) $([0, \alpha], [0, \beta]) \precsim_* ([0, \gamma], [0, \delta]) \Leftrightarrow \log(\beta) - \log(\alpha) \geq \log(\delta) - \log(\gamma)$ for all $\alpha, \beta, \gamma, \delta \in T$ with $\alpha, \gamma > 0$.

Fix any $(A|B, C|D), (A^*|B^*, C^*|D^*) \in \Theta_S \cup \Theta_T$, and set $a = \pi(A \cap B)/\pi(B)$, $a^* = \pi(A^* \cap B^*)/\pi(B^*)$, $b = \pi(C \cap D)/\pi(D)$, $b^* = \pi(C^* \cap D^*)/\pi(D^*)$. Then we have that

$$\pi(A \cap B)/\pi(B) = \pi([0, a])/1,\ \pi(C \cap D)/\pi(D) = \pi([0, b])/1,$$
$$\pi(A^* \cap B^*)/\pi(B^*) = \pi([0, a^*])/1,\ \pi(C^* \cap D^*)/\pi(D^*) = \pi([0, b^*])/1 . \tag{10}$$

It holds by Theorem 2 that

$$A|B \sim'' [0, a]|[0, 1],\ C|D \sim'' [0, b]|[0, 1],$$
$$A^*|B^* \sim'' [0, a^*]|[0, 1],\ C^*|D^* \sim'' [0, b^*]|[0, 1]. \tag{11}$$

It holds by (11), $\mathbf{L_9}$, (9), Lemma 5(iii) and (10) that

$$(A|B, C|D) \precsim (A^*|B^*, C^*|D^*) \Leftrightarrow ([0, a]|[0, 1], [0, b]|[0, 1]) \precsim ([0, a^*]|[0, 1], [0, b^*]|[0, 1])$$

$$\Leftrightarrow ([0, a], [0, b]) \precsim_* ([0, a^*], [0, b^*]) \Leftrightarrow \log\pi([0, b]) - \log\pi([0, a]) \geq \log\pi([0, b^*]) - \log\pi([0, a^*]).$$

$$\Leftrightarrow \log[\pi(C \cap D)/\pi(D)] - \log[\pi(A \cap B)/\pi(B)] \geq \log[\pi(C^* \cap D^*)/\pi(D^*)] - \log[\pi(A^* \cap B^*)/\pi(B^*)]. \quad \square$$

**Proof of Theorem 3 (ii)**: Let $F$ be a real-valued function on $\Gamma_S \cup \Gamma_T$. If there exists $a > 0$ and $b$ such that $F(A|B) = a \cdot F^1(A|B) + b$ for all $A|B \in \Gamma_S \cup \Gamma_T$, then $F$ is a likelihood function representing the relation $\succsim$.

Conversely, suppose that $F$ is a likelihood function representing the relation $\succsim$. We will prove that there exists $a > 0$ and $b$ such that $F(A|B) = a \cdot F^1(A|B) + b$ for all $A|B \in \Gamma_S \cup \Gamma_T$. We need a lemma:

**Lemma 6**: Let $g : [0, 1] \to \mathbb{R} \cup \{-\infty, +\infty\}$ be a function such that $g(\beta) - g(\alpha) \geq g(\delta) - g(\gamma) \Leftrightarrow \log(\beta) - \log(\alpha) \geq \log(\delta) - \log(\gamma)$ for all $\alpha, \beta, \gamma, \delta \in [0, 1]$ with $\alpha, \gamma > 0$. The following assertions hold: (**i**) $g$ is strictly increasing on $[0, 1]$ and $g(1) < +\infty$. (**ii**) Letting $f : (-\infty, 0] \to \mathbb{R}$ be a function defined by $f(x) = g(e^x)$ for all $x \in (-\infty, 0]$, it holds that $y - x = w - z \Leftrightarrow f(y) - f(x) = f(w) - f(z)$ for all $x, y, z, w \in (-\infty, 0]$. (**iii**) $f$ is strictly increasing on $(-\infty, 0]$. (**iv**) $f$ is continuous on $(-\infty, 0]$. (**v**) $f(q/p) = (q/p) \cdot [\, f(0) - f(-1)\, ] + f(0)$ for all integers $q > 0$ and $p < 0$. (**vi**) $g(\lambda) = a \cdot \lambda + b$ for all $\lambda \in [0, 1]$, where $a = f(0) - f(-1) > 0$ and $b = f(0)$.

Setting $g(t) = F([0, t]|[0, 1])$ and $F^1([0, t]|[0, 1]) = \log t$ for all $t \in [0, 1]$, it holds by Lemma 6 that there exists $a > 0$ and $b$ such that

$$F([0, t]|[0, 1]) = a \cdot F^1([0, t]|[0, 1]) + b \text{ for all } t \in [0, 1]. \tag{12}$$

Fix any $A|B \in \Gamma_S \cup \Gamma_T$. Setting $\lambda = \pi(A \cap B)/\pi(B)$, it holds by Theorem 2 that $[0, \lambda]|[0, 1] \sim'' A|B$ and

$$F^1([0, \lambda]|[0, 1]) = F^1(A|B). \tag{13}$$

Because $([0, \lambda]|[0, 1], [0, 1]|[0, 1]) \sim (A|B, \tau(B)|\tau(B))$ by $\log(\lambda/1) - \log 1 = \log(\lambda/1) - \log 1$, and because $F(T|T) = F(S|S)$ by $F(\phi_S|S) = F(\phi_T|T)$ and $(\phi_T|T, T|T) \sim (\phi_S|S, S|S)$, we have that $F([0, \lambda]|[0, 1]) = F(A|B)$. Thus we have by this, (12) and (13) that $F(A|B) = F([0, \lambda]|[0, 1]) = a \cdot F^1([0, \lambda]|[0, 1]) + b = a \cdot F^1(A|B) + b$.

Suppose that $F^*$ and $F$ represent $\succsim$. It holds by the above arguments that there exists $a > 0$ and $b$ such that

$$F^*(A|B) = a \cdot F^1(A|B) + b \text{ for all } A|B \in \Gamma_S \cup \Gamma_T,$$

and that there exists $a^* > 0$ and $b^*$ such that

$$F(A|B) = a^* \cdot F^1(A|B) + b^* \quad \text{for all } A|B \in \Gamma_S \cup \Gamma_T.$$

Hence it holds that $F(A|B) = a^* \cdot F^1(A|B) + b^* = a^* \cdot [F^*(A|B) - b]/a + b^* = (a/a^*) \cdot F^*(A|B) + [b^* - (a^* b)/a]$ for all $A|B \in \Gamma_S \cup \Gamma_T$, $\square$

**Proof of Theorem 4** (**i**) : Suppose that $\precsim$ is represented by a logarithmic likelihood function with respect to the probability function $\pi$ on $\mathcal{B}_S \cup \mathcal{B}_T$ satisfying the condition in Theorem 2. It holds by Lemma 3 and Theorem 2 that $\succsim$ satisfies $\mathbf{L_1} - \mathbf{L_8}$. Moreover, we can prove easily that $\succsim$ satisfies $\mathbf{L_9} - \mathbf{L_{10}}$ and $\mathbf{L^*_{11}}$ .

Conversely, suppose that $\succsim$ satisfies all the axioms. We will show that $\succsim$ is represented by a linear likelihood function with respect to a probability function $\pi$ on $\mathcal{B}_S \cup \mathcal{B}_T$ satisfying the condition in Theorem 2. We need a lemma:

**Lemma 7**: (**i**) If $1 \geq \alpha \geq \beta > 0$, then $([0, \beta], [0, \beta]) \sim_* ([0, \alpha], [0, \alpha])$. (**ii**) $1 \geq \alpha \geq \beta > 0 \Leftrightarrow ([0, \beta], [0, \alpha]) \succsim_* ([0, \beta], [0, \beta])$. (**iii**) If $1 \geq \alpha > \beta > 0$, then $([0, \beta], [0, \alpha]) \succ_* ([0, \beta], [0, \beta])$. (**iv**) $([0, \alpha], [0, \beta]) \succsim_* ([0, \gamma], [0, \delta]) \Leftrightarrow \beta - \alpha \geq \delta - \gamma$ for all $\alpha, \beta, \gamma, \delta \in T$ with $\alpha > 0, \gamma > 0$.

Fix any $(A|B, C|D), (A^*|B^*, C^*|D^*) \in \Theta_S \cup \Theta_T$, and set $a = \pi(A \cap B)/\pi(B)$, $a^* = \pi(A^* \cap B^*)/\pi(B^*)$, $b = \pi(C \cap D)/\pi(D)$, $b^* = \pi(C^* \cap D^*)/\pi(D^*)$. Then we have that

$$\pi(A \cap B)/\pi(B) = \pi([0, a])/1,\ \pi(C \cap D)/\pi(D) = \pi([0, b])/1,$$
$$\pi(A^* \cap B^*)/\pi(B^*) = \pi([0, a^*])/1,\ \pi(C^* \cap D^*)/\pi(D^*) = \pi([0, b^*])/1 . \tag{14}$$

It holds by Theorem 2 that

$$A|B \sim'' [0, a]|[0, 1],\ C|D \sim'' [0, b]|[0, 1],$$
$$A^*|B^* \sim'' [0, a^*]|[0, 1],\ C^*|D^* \sim'' [0, b^*]|[0, 1]. \tag{15}$$

It holds by (15), $\mathbf{L_9}$, (9), Lemma 7(iv) and (14) that

$$(A|B, C|D) \succsim (A^*|B^*, C^*|D^*) \Leftrightarrow ([0, a]|[0, 1], [0, b]|[0, 1]) \succsim ([0, a^*]|[0, 1], [0, b^*]|[0, 1])$$
$$\Leftrightarrow ([0, a], [0, b]) \succsim_* ([0, a^*], [0, b^*]) \Leftrightarrow \pi([0, a]) - \pi([0, b]) \geq \pi([0, a^*]) - \pi([0, b^*])$$
$$\Leftrightarrow \pi(A \cap B)/\pi(B) - \pi(C \cap D)/\pi(D) \geq \pi(A^* \cap B^*)/\pi(B^*) - \pi(C^* \cap D^*)/\pi(D^*). \quad \square$$

**Proof of Theorem 4** (**ii**) : Let $F$ be a real-valued function on $\Gamma_S \cup \Gamma_T$. If there exists $a > 0$ and $b$ such that $F(A|B) = a \cdot F^2(A|B) + b$ for all $A|B \in \Gamma_S \cup \Gamma_T$, then $F$ is a likelihood function representing the relation $\precsim$. Conversely, suppose that $F$ is a likelihood function representing the relation $\precsim$. We will prove that there exists $a > 0$ and $b$ such that $F(A|B) = a \cdot F^2(A|B) + b$ for all $A|B \in \Gamma_S \cup \Gamma_T$.

**Lemma 8**: Let $g : [0, 1] \to \mathbb{R} \cup \{-\infty, +\infty\}$ be a function such that

$$g(\beta) - g(\alpha) \geq g(\delta) - g(\gamma) \Leftrightarrow \beta - \alpha \geq \delta - \gamma \text{ for all } \alpha, \beta, \gamma, \delta \in [0, 1].$$

It holds that: (**i**) $g$ is strictly increasing on $[0, 1]$. (**ii**) $g$ is continuous on $[0, 1]$. (**iii**) For any positve integer $p > 0$, it holds that $g(q/p) = (q/p) \cdot [\, g(1) - g(0)\,] + g(0)$ for all $q = 0, 1, 2, \cdots, p$. (**iv**) $g(\lambda) = a \cdot \lambda + b$ for all $\lambda \in [0, 1]$, where $a = g(1) - g(0) > 0$ and $b = g(0)$.

Setting $g(t) = F([0, t]|T)$ and $h(t) = F^2([0, t]|T) = t$ for all $t \in [0, 1]$, it holds by Lemma 8(iii) that there exists $a > 0$ and $b$ such that

$$F([0, t]|T) = a \cdot F^2([0, t]|T) + b \text{ for all } t \in [0, 1]. \tag{16}$$

Fix any $A \in \mathcal{B}$. It holds by Lemma 1(iv) that there is $\lambda \in [0, 1]$ such that $[0, \lambda] \sim' A$, which implies that $F([0, \lambda]) = F(A)$ and $F^*([0, \lambda]) = F^*(A)$. Thus we have by (16) that

$$F(A) = F([0, \lambda]) = a \cdot F^2([0, \lambda]) + b = a \cdot F^*(A) + b. \qquad \square$$

## Appendix

**Proof of Lemma 1**: (**i**) It holds by $\mathbf{L_4}$ that $\{a\} = [a, a] \sim' \phi$ for all $a \in [0, 1]$. (**ii**) Fix any $a, b \in [0, 1]$ with $b > a$, it holds by $\mathbf{L_4}$ that $[0, b - a] \succ' \phi$. It holds by $\mathbf{L_5}$ that $[a, b] \succ' \phi$. (**iii**) Fix any $a, b \in [0, 1]$ with $a < b$. Because $[a, b) \precsim' [a, b)$ and $\{b\} \precsim' \phi$ by Lemma 1(i) it holds by $\mathbf{L_2}$ that $[a, b] \precsim' [a, b)$. Because $\phi \precsim' \{b\}$ by Lemma 1(i), it holds by $[a, b) \precsim' [a, b)$ and $\mathbf{L_2}$ that $[a, b) \precsim' [a, b]$. Hence $[a, b] \sim' [a, b)$. By almost the same manner we can prove that $[a, b] \sim' (a, b] \sim' (a, b)$. (**iv**) Suppose $a \geq b$. Because $[0, b) \precsim' [0, b)$ and $[b, a] \precsim' \phi$ by $\mathbf{L_4}$, it holds by $\mathbf{L_2}$ that $[0, a] \precsim' [0, b)$. It holds by Lemma 1(i) that $[0, a] \precsim' [0, b) \sim' [0, b]$. Suppose $b > a$. Because $[0, a) \precsim' [0, a)$ and $[a, b] \succ' \phi$ by Lemma 1(ii), it holds by $\mathbf{L_2}$ that $[0, b] \succ' [0, a)$. It holds by

Lemma 1(iii) that $[0, b] \succ' [0, a) \sim' [0, a]$. Hence we have that $b > a \Rightarrow [0, b] \succ' [0, a]$, which implies $[0, a] \succsim' [0, b] \Rightarrow a \geq b$. (**v**) It follows from Lemma 1(i) that it suffices to prove the case of the closed intervals. For any intervals $[a, b], [c, d] \in \mathcal{I}_T$, it holds by $\mathbf{L_5}$ that $[0, b-a] \sim' [a, b]$, $[0, d-c] \sim^* [c, d]$. Hence we have by Lemma 1(iv) that $m([a, b]) \geq m([c, d]) \Leftrightarrow (b-a) \geq (d-c) \Leftrightarrow [0, b-a] \succsim' [0, d-c] \Leftrightarrow [a, b] \succsim' [c, d]$. (**vi**) Suppose that $\tau(A) - A \succ' \tau(B) - B$. It holds by $A \succsim' B$ and $\mathbf{L_2}$ that $\tau(A) \succ' \tau(B)$, which is a contradiction. Thus we have that $\tau(B) - B \succsim' \tau(A) - A$. (**vii**) Suppose that $B_n \subset B_{n+1}$ for all $n$ and that there exists $A \in \mathcal{B}_S \cup \mathcal{B}_T$ such that $A \succsim' B_n$ for all $n$. Define $C_n = \tau(B_n) - B_n$ for all $n$, and define $D = \tau(A) - A$. Then it holds by $B_n \subset B_{n+1}$ that $C_n \supset C_{n+1}$ for all $n$ and it holds by Lemma 1(vi) that $C_n \succsim' D$ for all $n$. Hence we have by $\mathbf{L_3}$ that $\cap C_n \succsim' D$. By this and Billingsley (1995, Problem 2.1, page 32), we have that $A = \tau(D) - D \succsim' \tau(\cap C_n) - (\cap C_n) = \cup[\tau(C_n) - C_n] = \cup B_n$, because $\tau(C_n) - C_n = B_n$ for all $n$. (**viii**) Define $\{B_n\}$ in $\mathcal{B}_T$ by $B_n = [0, x_n]$ for all $n$. We have $B_n \supset B_{n+1}$ for all $n$ by $x_n \geq x_{n+1}$ for all $n$. Because $B_n \succsim' A$ for all $n$, we have by $\mathbf{L_3}$ that $[0, \lim x_n] = \cap B_n \succsim' A$. (**ix**) Define $\{B_n\}$ in $\mathcal{B}_T$ by $B_n = [0, x_n]$ for all $n$. We have $B_n \subset B_{n+1}$ for all $n$ by $x_n \leq x_{n+1}$ for all $n$. Since $A \succsim' B_n$ for all $n$, we have by Lemma 1(vii) that $A \succsim' \cup B_n = [0, \lim x_n)$. It holds by Lemma 1(i) that $[0, \lim x_n] \sim' [0, \lim x_n)$. Thus we have $A \succsim' [0, \lim x_n]$. (**x**) Fix any $A \in \mathcal{B}_S \cup \mathcal{B}_T$. It holds by $\mathbf{L_1}$(ii) that $0 \in \{x \in T : A \succsim' [0, x]\} \neq \phi$. Because $A \succsim' A$ and $\tau(A) - A \succsim' \phi$ by $\mathbf{L_1}$(ii), we have by $\mathbf{L_2}$ that $T \succsim' A$, which implies that $1 \in \{x \in T : [0, x] \succsim' A\} \neq \phi$. Let $\{x_n\}$ be a sequence in $\{x \in T : A \succsim' [0, x]\}$ converging to $x^*$. It holds by Thurston (1994) that $\{x_n\}$ has a subsequence $\{y_n\}$ converging to $x^*$ satisfying (**a**) $y_n \leq y_{n+1}$ for all $n$, or (**b**) $y_n \geq y_{n+1}$ for all $n$. In the case of (**a**), it holds by Lemma 1(ix) that $A \succsim' [0, x^*]$. In the case of (**b**), it holds by Lemma 1(iv) that $A \succsim' [0, x^*]$. Hence $\{x \in T : A \succsim' [0, x]\}$ is closed in T. By almost the same manner, we can prove that $\{x \in T : [0, x] \succsim' A\}$ are closed in T, using Lemma 1(iv, viii). □

**Proof of Lemma 2**: (**i**) It holds by $\mathbf{L_6}$(ii) that $A_2 | A_1 \sim'' B_2 | B_1$ and $A_4 | A_3 \sim'' B_4 | B_3$. Thus we have that $A_2 | A_1 \succsim'' A_4 | A_3 \Leftrightarrow B_2 | B_1 \succsim'' B_4 | B_3$.

(**ii**) **Case 1** ($\alpha \geq \gamma$): It holds by $1 \geq (\gamma/\alpha) > 0$ and $\mathbf{L_7}$ that $[0, \beta] | [0, \alpha] \sim'' [0, (\gamma/\alpha)\beta] | [0, (\gamma/\alpha)\alpha] =$

$[0, (\gamma/\alpha)\beta] \mid [0, \gamma]$. Hence we have by $\mathbf{L_6}$(i) and Theorem 1(ii) that

$$[0, \beta] \mid [0, \alpha] \precsim'' [0, \delta] \mid [0, \gamma] \Leftrightarrow [0, (\gamma/\alpha)\beta] \mid [0, \gamma] \precsim'' [0, \delta] \mid [0, \gamma]$$

$$\Leftrightarrow [0, (\gamma/\alpha)\beta] \precsim' [0, \delta] \Leftrightarrow (\gamma/\alpha)\beta \geq \delta \Leftrightarrow \beta/\alpha \geq \delta/\gamma.$$

**Case 2** ($\alpha < \gamma$): It holds by $0 < \alpha/\gamma < 1$ and $\mathbf{L_7}$ that $[0, \delta] \mid [0, \gamma] \sim'' [0, (\alpha/\gamma)\delta] \mid [0, (\alpha/\gamma)\gamma] = [0, (\alpha/\gamma)\delta] \mid [0, \alpha]$. Hence we have by $\mathbf{L_6}$(i) and Theorem 1(ii) that

$$[0, \beta] \mid [0, \alpha] \precsim'' [0, \delta] \mid [0, \gamma] \Leftrightarrow [0, \beta] \mid [0, \alpha] \precsim'' [0, (\alpha/\gamma)\delta] \mid [0, \alpha]$$

$$\Leftrightarrow [0, \beta] \precsim' [0, (\alpha/\gamma)\delta] \Leftrightarrow \beta \geq (\alpha/\gamma)\delta \Leftrightarrow \beta/\alpha \geq \delta/\gamma.$$

(**iii**): Fix any $A \mid B, C \mid D \in \Gamma_S \cup \Gamma_T$ with $A \subset B$ and $C \subset D$. It holds by Theorem 1(ii), Lemma 2(i, ii) that $A \mid B \precsim'' C \mid D \Leftrightarrow [0, \pi(A)] \mid [0, \pi(B)] \precsim'' [0, \pi(C)] \mid [0, \pi(D)] \Leftrightarrow \pi(A)/\pi(B) \geq \pi(C)/\pi(D)$. □

**Proof of Lemma 3**: Suppose that there is a likelihood function $F$ representing a relative likelihood relation $\precsim$, and that $F$ is logarithmic or linear with respect to a probability function $\pi$. It holds by (1), (3), (4) and (6) that

$$A \mid B \precsim'' C \mid D \Leftrightarrow (\tau(B) \mid \tau(B), A \mid B) \precsim (\tau(D) \mid \tau(D), C \mid D) \Leftrightarrow \pi(A \cap B)/\pi(B) \geq \pi(C \cap D)/\pi(D)$$

for all $A \mid B, C \mid D \in \Gamma_S \cup \Gamma_T$. □

**Proof of Lemma 4**: (**i**) It holds by (5) that $A \mid T \sim'' A^* \mid T$ and $B \mid T \sim'' B^* \mid T$. It holds by $\mathbf{L_9}$ that $(A \mid T, B \mid T) \sim (A^* \mid T, B^* \mid T)$. (**ii**) Fix any $(A, B), (C, D) \in \Delta_T$ with $(B, A), (D, C) \in \Delta_T$. It holds by $\mathbf{L_{10}}$ that $(A, B) \precsim_* (C, D) \Rightarrow (D, C) \precsim_* (B, A)$. By the contraposition of this, we have that $(B, A) \succ_* (D, C) \Rightarrow (C, D) \succ_* (A, B)$. □

**Proof of Lemma 5**: (**i**) Suppose that $A \subset B \succ' \phi$ and $C \subset D \succ' \phi$. It holds by (6) that

$$A \mid B \precsim'' C \mid D \Leftrightarrow (T \mid T, A \mid B) \precsim (T \mid T, C \mid D). \tag{17}$$

and it holds by Theorem 2 and $\mathbf{L_9}$ that

$$(T \mid T, A \mid B) \sim (B \mid B, A \mid B) \text{ and } (T \mid T, C \mid D) \sim (D \mid D, C \mid D).$$

Hence we have by (17) and this that

$$A \mid B \precsim'' C \mid D \Leftrightarrow (B \mid B, A \mid B) \precsim (D \mid D, C \mid D). \tag{18}$$

It holds by $\mathbf{L_{11}}$ and $A \subset B \succ' \phi$ that

$(B \mid B, A \mid B) \sim (B \mid T, A \mid T)$ and $(D \mid D, C \mid D) \sim (D \mid T, C \mid T)$.

Hence we have by this, (18) and (9) that

$$A \mid B \precsim'' C \mid D \iff (B \mid T, A \mid T) \precsim (D \mid T, C \mid T) \iff (B, A) \precsim_* (D, C) .$$

(**ii**) The axiom $\mathbf{L_7}$ and Lemma 5(i) together imply Lemma 5(ii).

(**iii**) Fix $\alpha, \beta, \gamma, \delta \in T$ with $\alpha > 0, \gamma > 0$.

**Case 1** ($\beta \geq \alpha$ and $\delta \geq \gamma$): It holds by Lemma 5(ii) that

$$([0, \alpha/\beta], [0, 1]) \sim_* ([0, \alpha], [0, \beta]) \quad \text{and} \quad ([0, \gamma/\delta], [0, 1]) \sim_* ([0, \gamma], [0, \delta]) .$$

We have by this, Lemma 4(i), (6), $\mathbf{L_{10}}$ and Theorem 1(ii) that

$$([0, \alpha], [0, \beta]) \precsim ([0, \gamma], [0, \delta]) \iff ([0, \alpha/\beta], [0, 1]) \precsim_* ([0, \gamma/\delta], [0, 1])$$

$$\iff ([0, 1], [0, \gamma/\delta]) \precsim_* ([0, 1], [0, \alpha/\beta]) \iff [0, \gamma/\delta] \precsim' [0, \alpha/\beta]$$

$$\iff \gamma/\delta \geq \alpha/\beta \iff \beta/\alpha \geq \delta/\gamma \iff \log(\beta) - \log(\alpha) \geq \log(\delta) - \log(\gamma).$$

**Case 2** ($\beta \geq \alpha$ and $\delta < \gamma$): We show that $([0, \alpha], [0, \beta]) \precsim_* ([0, \gamma], [0, \delta]) \succ_* ([0, \alpha], [0, \beta])$ and $\log(\beta) - \log(\alpha) \geq \log(\delta) - \log(\gamma)$ hold simultaneously. It holds by $\beta \geq \alpha$ and $\delta < \gamma$ that $\log(\beta) - \log(\alpha) \geq \log(\delta) - \log(\gamma)$. We prove that $([0, \alpha], [0, \beta]) \precsim_* ([0, \gamma], [0, \delta])$. It holds by Lemma 5(ii) that $([0, 1], [0, \delta/\gamma]) \sim_* ([0, \gamma], [0, \delta])$. We have by this and Theorem 1(ii) that

$$([0, 1], [0, 1]) \precsim_* ([0, 1], [0, \delta/\gamma]) \sim_* ([0, \gamma], [0, \delta]) \tag{19}$$

It holds by Lemma 5(ii) that $([0, \alpha/\beta], [0, 1]) \sim_* ([0, \alpha], [0, \beta])$. We have by this and $\mathbf{L_{10}}$ that

$$([0, 1], [0, \alpha/\beta]) \sim_* ([0, \beta], [0, \alpha]).$$

We have by Theorem 1(ii) and this that $([0, 1], [0, 1]) \precsim_* ([0, 1], [0, \alpha/\beta]) \sim_* ([0, \beta], [0, \alpha])$. Hence we have by $\mathbf{L_{10}}$ that

$$([0, \alpha], [0, \beta]) \precsim_* ([0, 1], [0, 1]). \tag{20}$$

Thus we have by (19) and (20) that $([0, \alpha], [0, \beta]) \precsim_* ([0, \gamma], [0, \delta])$.

**Case 3** ($\beta < \alpha$ and $\delta < \gamma$): We have by $\mathbf{L_{10}}$ and Case 1 that

$$([0, \alpha], [0, \beta]) \precsim_* ([0, \gamma], [0, \delta]) \iff ([0, \delta], [0, \gamma]) \precsim_* ([0, \beta], [0, \alpha]) \iff \gamma/\delta \geq \alpha/\beta$$

$$\iff \beta/\alpha \geq \delta/\gamma \iff \log(\beta) - \log(\alpha) \geq \log(\delta) - \log(\gamma).$$

**Case 4** ($\beta < \alpha$ and $\delta \geq \gamma$): Applying the logical equivalence: $(P \Leftrightarrow Q) \equiv$ (**not** $P \Leftrightarrow$ **not** $Q$), it suffices to prove that $([0, \gamma], [0, \delta]) \succ_* ([0, \alpha], [0, \beta]) \iff \log(\delta) - \log(\gamma) > \log(\beta) - \log(\alpha)$. We show that

$([0,\ \gamma], [0, \delta]) \succ_* ([0, \alpha], [0, \beta])$ and $\log(\delta) - \log(\gamma) > \log(\beta) - \log(\alpha)$ hold independently in this case. It holds by $\beta < \alpha$ and $\delta \geq \gamma$ that $\log(\delta) - \log(\gamma) > \log(\beta) - \log(\alpha)$. There remains to prove that $([0,\ \gamma], [0, \delta]) \succ_* ([0, \alpha], [0, \beta])$. Suppose that

$$([0, \alpha], [0, \beta]) \succsim_* ([0,\ \gamma], [0, \delta]). \tag{21}$$

Because $\alpha/\alpha = 1 > \beta/\alpha$, it holds by Lemma 5(i) and Theorem 2 that

$$([0, \alpha], [0, \alpha]) \succ_* ([0, \alpha], [0, \beta]). \tag{22}$$

Because $\delta/\gamma \geq 1 = \gamma/\gamma$, it holds by Lemma 5(i) and Theorem 2 that $([0,\ \gamma], [0, \delta]) \succsim_* ([0,\ \gamma], [0, \gamma])$. Hence it holds by $\gamma/\gamma = \alpha/\alpha$, Lemma 5(i) and Theorem 2 that

$$([0,\ \gamma], [0, \delta]) \succsim_* ([0,\ \gamma], [0, \gamma]) \sim_* ([0, \alpha], [0, \alpha]). \tag{23}$$

We have by (21), (22) and (23) that $([0,\alpha], [0,\alpha]) \succ_* ([0,\alpha], [0,\beta]) \succ_* ([0,\alpha],[0,\beta]) \succsim_* ([0,\ \gamma], [0, \delta]) \sim_* ([0,\alpha], [0,\alpha])$. This is a contradiction. Hence $([0, \gamma], [0, \delta]) \succ_* ([0, \alpha], [0, \beta])$. □

**Proof of Lemma 6**: (**i**) It holds by the supposition of Lemma 6 that $g(\beta) - g(\alpha) \geq g(1) - g(1) \Leftrightarrow \log \beta - \log \alpha \geq \log 1 - \log 1$ for all $\alpha, \beta \in (0, 1]$, which implies that $g(\beta) - g(\alpha) \geq 0 \Leftrightarrow \log(\beta/\alpha) \geq \log 1$ and $g(\beta) \geq g(\alpha) \Leftrightarrow \beta \geq \alpha$. Hence $g$ is strictly increasing on $(0, 1]$.

If $g(0) \geq g(\lambda^*)$ for some $\lambda^* \in (0, 1]$, then $g(0) - g(1) \geq g(\lambda^*) - g(1)$. On the other hand, we have by the supposition of Lemma 6 and (2) that

$$\log(\lambda^*) > \log(0) \Rightarrow \log(\lambda^*) - \log(1) > \log(0) - \log(1) \Rightarrow g(\lambda^*) - g(1) > g(0) - g(1).$$

This is a contradiction. Thus $g(0) < g(\lambda)$ for all $\lambda \in (0, 1]$ and $g$ is strictly increasing on $[0, 1]$. Moreover , it holds that

$$\log(1/2) > \log(1/4) \Rightarrow \log(1/2) - \log(1) > \log(1/8) - \log(1/2)$$
$$\Rightarrow g(1/2) - g(1) > g(1/8) - g(1/2) \Rightarrow g(1/2) - g(1/8) + g(1/2) > g(1).$$

Because $g$ is strictly increasing on $[0, 1]$, we have that $g(1) < +\infty$.

(**ii**) It holds that

$$f(\log \lambda) = g(\lambda) \quad \text{for all } \lambda \in (0, 1]. \tag{24}$$

It holds by the supposition of Lemma 6 that

$$\log \beta - \log \alpha = \log \delta - \log \gamma \Leftrightarrow g(\beta) - g(\alpha) = g(\delta) - g(\gamma) \tag{25}$$

for all $\alpha, \beta, \gamma, \delta \in (0, 1]$. For all $x, y, z, w \in (-\infty, 0]$, set $\alpha = e^x$, $\beta = e^y$, $\gamma = e^z$ and $\delta = e^w$. Then we have by (24) and (25) that $y - x = w - z \Leftrightarrow f(y) - f(x) = f(w) - f(z)$ for all $x, y, z, w \in (-\infty, 0]$.

(**iii**) Because $g$ is strictly increasing on $[0, 1]$ by Lemma 6(i), and because $e^x$ is strictly increasing on $(-\infty, 0]$, $f(x) = g(e^x)$ is strictly increasing for $x \in (-\infty, 0]$.

(**iv**) It holds by Lemma 6(iii) and Royden and Fitzpatrick (2010, Section 6.1, Theorem 1) that there are at most countable number of points at which $f$ is not continuous, and then there is a point $x$ in $(-\infty, 0)$ at which $f$ is continuous. Let $y$ be a point in $(-\infty, 0]$, and let $\{ y_m \}$ be a convergent sequence in $(-\infty, 0]$ to $y$. Define a sequence $\{ x_m \}$ by $x_m = x - y + y_m$ for all $m$. Because $-x > 0$, there exists some integer $m^*$ such that $-y + y_m < -x$ for all $m \geq m^*$, which implies that $x_m \in (-\infty, 0]$ for all $m \geq m^*$. Hence we have by Lemma 6(ii) that $f(x_m) - f(x) = f(y_m) - f(y)$ for all $m \geq m^*$. Because $\lim y_m = y$ and $f$ is continuous at $x$, we have that $\lim f(y_m) = f(y)$, and that $f(\cdot)$ is continuous on $(-\infty, 0]$.

(**v**) Using the induction arguments with respect to $q = 0, 1, 2, \cdots$ for a fixed negative integer $p < 0$, it holds by Lemma 6(ii) that

$$f(q/p) = [\, f(1/p) - f(0) \,]\cdot q + f(0) \text{ for all integers } q \geq 0 \text{ and } p < 0. \tag{26}$$

For each $p < 0$, setting $q = -p$ in (26), we have that

$$f(-1) = -[\, f(1/p) - f(0) \,]\cdot p + f(0) \text{ and } f(1/p) = [\, f(0) - f(-1) \,]/p + f(0) \text{ for all } p < 0.$$

It holds by (26) and this that $f(q/p) = (q/p)\cdot[\, f(0) - f(-1) \,] + f(0)$ for all integers $q \geq 0$ and $p < 0$.

(**vi**) Fix any rational number $r$ in $(-\infty, 0]$. There exists a pair of integers $(p^*, q^*)$ such that

$$r = q^*/p^*,\ q^* \geq 0 \text{ and } p^* < 0.$$

Because $a = f(0) - f(-1)$ and $b = f(0)$, we have by Lermma 6(v) that

$$f(r) = f(q^*/p^*) = a\cdot(q^*/p^*) + b = a\cdot r + b \tag{27}$$

for all rational numbers $r$ in $(-\infty, 0]$. Because $f(x)$ is continuous on $(-\infty, 0]$ by Lemma 6(iv), we have by Lemma 6(v) and (27) that $f(x) = a\cdot x + b$ for all real numbers $x \in (-\infty, 0]$, which implies that

$$g(\lambda) = f(\log \lambda) = a\cdot \log \lambda + b \text{ for all } \lambda \in (0, 1]. \tag{28}$$

It holds by (28) that $\lim_{\lambda\to 0} g(\lambda) = -\infty$. Hence we have Lemma 6(i) and (2) that $g(0) = -\infty$.

Because $\log 0 = \lim_{\lambda \to 0} \log \lambda = -\infty$, we have by (2) that $g(0) = a \cdot \log 0 + b$. Thus we have by this and (27) that $g(\lambda) = a \cdot \log \lambda + b$ for all $\lambda \in [0, 1]$. □

**Proof of Lemma 7**: (**i**) Fix any $1 \geq \alpha \geq \beta > 0$. It holds by $\mathbf{L}_{11}^*$ that

$$([0, \beta], [0, \beta]) \sim_* ([0, \beta] \cup (\beta, \alpha], [0, \beta] \cup (\beta, \alpha]) = ([0, \alpha], [0, \alpha]).$$

(**ii**) It holds by $\mathbf{L}_{11}^*$, $\mathbf{L}_{10}$, (9) and (5) that

$$([0, \beta], [0, \alpha]) \precsim_* ([0, \beta], [0, \beta])$$

$$\Leftrightarrow ([0, \beta + (1-\alpha)], [0, 1]) \sim_* ([0, \beta], [0, \alpha]) \precsim_* ([0, \beta], [0, \beta]) \sim_* ([0, 1], [0, 1])$$

$$\Leftrightarrow ([0, 1], [0, 1]) \precsim_* ([0, 1], [0, \beta + (1-\alpha)])$$

$$\Leftrightarrow (T \mid T, [0, 1] \mid T) \precsim_* (T \mid T, [0, \beta + (1-\alpha)] \mid T)$$

$$\Leftrightarrow [0, 1] \precsim' [0, \beta + (1-\alpha)] \Leftrightarrow 1 \geq \beta + (1-\alpha) \Leftrightarrow \alpha \geq \beta.$$

(**iii**) It holds by Lemma 7(ii) that $([0, \beta], [0, \alpha]) \succ_* ([0, \beta], [0, \beta])$.

(**iv**) Fix $\alpha, \beta, \gamma, \delta \in T$ with $\alpha > 0, \gamma > 0$.

**Case 1** ($\beta \geq \alpha$ and $\delta \geq \gamma$): We have by Lemma 4(i), $\mathbf{L}_{10}$, $\mathbf{L}_{11}^*$ and Theorem 1(ii) that

$$([0, \alpha], [0, \beta]) \precsim_* ([0, \gamma], [0, \delta]) \Leftrightarrow ([0, \alpha + (1-\beta)], [0, 1]) \precsim_* ([0, \gamma + (1-\delta)], [0, 1])$$

$$\Leftrightarrow ([0, 1], [0, \gamma + (1-\delta)]) \precsim_* ([0, 1], [0, \alpha + (1-\beta)])$$

$$\Leftrightarrow [0, \gamma + (1-\delta)] \precsim' [0, \alpha + (1-\beta)] \Leftrightarrow \gamma + (1-\delta) \geq \alpha + (1-\beta) \Leftrightarrow \beta - \alpha \geq \delta - \gamma.$$

**Case 2** ($\beta \geq \alpha$ and $\delta < \gamma$): We show that $([0, \alpha], [0, \beta]) \precsim_* ([0, \gamma], [0, \delta])$ and $\beta - \alpha \geq \delta - \gamma$ hold simultaneously. We have by $\beta \geq \alpha$ and $\delta < \gamma$ that $\beta - \alpha \geq 0 > \delta - \gamma$. It holds by $\mathbf{L}_{11}^*$ that $([0, \gamma], [0, \delta]) \sim_* ([0, 1], [0, \delta + (1-\gamma)])$. We have by this and Theorem 1(ii) that

$$([0, 1], [0, 1]) \precsim_* ([0, 1], [0, \delta + (1-\gamma)]) \sim_* ([0, \gamma], [0, \delta]) \tag{29}$$

It holds by $\mathbf{L}_{11}{}^*$ that $([0, \alpha], [0, \beta]) \sim_* ([0, \alpha + (1-\beta)], [0, 1])$. We have by this and $\mathbf{L}_{10}$ that

$$([0, 1], [0, \alpha + (1-\beta)]) \sim_* ([0, \beta], [0, \alpha]).$$

We have by Theorem 1(ii) and this that $([0, 1], [0, 1]) \precsim_* ([0, 1], [0, \alpha + (1-\beta)]) \sim_* ([0, \beta], [0, \alpha])$. Hence we have by $\mathbf{L}_{10}$ that

$$([0, \alpha], [0, \beta]) \precsim_* ([0, 1], [0, 1]). \tag{30}$$

Thus we have by (29) and (30) that $([0, \alpha], [0, \beta]) \precsim_* ([0, \gamma], [0, \delta])$.

**Case 3** ($\beta < \alpha$ and $\delta < \gamma$): We have by $\mathbf{L_{10}}$ and Case 1 that

$$([0, \alpha], [0, \beta]) \succsim_* ([0, \gamma], [0, \delta]) \Leftrightarrow ([0, \delta], [0, \gamma]) \succsim_* ([0, \beta], [0, \alpha]) \Leftrightarrow \gamma - \delta \geq \alpha - \beta \Leftrightarrow \beta - \alpha \geq \delta - \gamma.$$

**Case 4** ($\beta < \alpha$ and $\delta \geq \gamma$): Applying the logical equivalence: $(P \Leftrightarrow Q) \equiv$ (**not** $P \Leftrightarrow$ **not** $Q$), it suffices to prove that $([0, \gamma], [0, \delta]) \succ_* ([0, \alpha], [0, \beta]) \Leftrightarrow \delta - \gamma > \beta - \alpha$. We show that $([0, \gamma], [0, \delta]) \succ_* ([0, \alpha], [0, \beta])$ and $\delta - \gamma > \beta - \alpha$ hold independently in this case. It holds by $\beta < \alpha$ and $\delta \geq \gamma$ that $\delta - \gamma > \beta - \alpha$. There remains to prove that $([0, \gamma], [0, \delta]) \succ_* ([0, \alpha], [0, \beta])$. Suppose that

$$([0, \alpha], [0, \beta]) \succsim_* ([0, \gamma], [0, \delta]). \tag{31}$$

It holds by $\beta < \alpha$ and Lemma 7(iii) that $([0, \beta], [0, \alpha]) \succ_* ([0, \beta], [0, \beta])$. Hence we have by $\mathbf{L_{10}}$ that

$$([0, \beta], [0, \beta]) \succ_* ([0, \alpha], [0, \beta]). \tag{32}$$

It holds by Lemma 7(ii) that $([0, \gamma], [0, \delta]) \succsim_* ([0, \gamma], [0, \gamma])$. It holds by this and Lemma 7(i) that

$$([0, \gamma], [0, \delta]) \succsim_* ([0, \gamma], [0, \gamma]) \sim_* ([0, \beta], [0, \beta]). \tag{33}$$

We have by (31), (32) and (33) that $([0, \beta], [0, \beta]) \succ_* ([0,\alpha], [0, \beta]) \succsim_* ([0, \gamma], [0, \delta]) \sim_* ([0,\beta], [0, \beta])$.

This is a contradiction. Thus $([0, \gamma], [0, \delta]) \succ_* ([0, \alpha], [0, \beta])$. □

**Proof of Lemma 8**: (**i**) It holds by the supposition of Lemma 8 that $g(\beta) - g(\alpha) \geq g(1) - g(1) \Leftrightarrow \beta - \alpha \geq 1 - 1$ for all $\alpha, \beta \in [0, 1]$, which implies that $g(\beta) \geq g(\alpha) \Leftrightarrow \beta \geq \alpha$. Hence $g$ is strictly increasing on $[0, 1]$.

(**ii**) It holds by the supposition of Lemma 8 that

$$\beta - \alpha = \delta - \gamma \Leftrightarrow g(\beta) - g(\alpha) = g(\delta) - g(\gamma) \tag{34}$$

for all $\alpha, \beta, \gamma, \delta \in [0, 1]$. It holds by Lemma 8(i) and Royden and Fitzpatrick (2010, Section 6.1, Theorem 1) that there are at most countable number of points at which $g$ is not continuous, and then there is a point $\lambda$ in $(0, 1)$ at which $g$ is continuous. Let $\mu$ be a point in $[0, 1]$, and let $\{\mu_m\}$ be a convergent sequence in $[0, 1]$ to $\mu$. Define a sequence $\{\lambda_m\}$ by $\lambda_m = \lambda + \mu_m - \mu$ for all $m$. Because $-\lambda < 0 < 1 - \lambda$, there exists some integer $m^*$ such that $-\lambda < \mu_m - \mu \leq 1 - \lambda$ for all $m \geq m^*$, which implies that $\lambda_m \in [0, 1]$ for all $m \geq m^*$. Hence we have by (34) that $g(\lambda_m) -$

$g(\lambda) = g(\mu_m) - g(\mu)$ for all $m \geq m^*$. Because $\lim \mu_m = \mu$ and $g$ is continuous at $\lambda$, we have that $\lim g(\mu_m) = g(\mu)$, and that $g(\cdot)$ is continuous on $[0, 1]$.

(**iii**) For each $p > 0$, using the induction arguments with respect to $q = 0, 1, 2, \cdots, p$, it holds by (34) that

$$g(q/p) = [\, g(1/p) - g(0)\, ]\cdot q + g(0) \quad \text{for all } q = 0, 1, 2, \cdots, p. \tag{35}$$

For each $p > 0$, setting $q = p$ in (35), we have that

$$g(1) = [\, g(1/p) - g(0)\, ]\cdot p + g(0) \text{ and } g(1/p) = [\, g(1) - g(0)\, ]/p + g(0) \text{ for all } p > 0.$$

It holds by (35) and this that

$$g(q/p) = (q/p)\cdot[\, g(1) - g(0)\, ] + g(0) \quad \text{for all } p > 0 \text{ and all } q = 0, 1, 2, \cdots, p.$$

(**iv**) Fix any rational number $r$ in $[0, 1]$. There exists a pair of integers $(p^*, q^*)$ such that

$$r = q^*/p^*, \; p^* > 0 \text{ and } p^* \geq q^* \geq 0.$$

Because $a = g(1) - g(0)$ and $b = g(0)$, we have by Lemma 8(iii) that

$$g(r) = g(q^*/p^*) = a\cdot(q^*/p^*) + b = a\cdot r + b \tag{36}$$

for all rational numbers $r$ in $[0, 1]$. Because $g$ is continuous on $[0, 1]$ by Lemma 8(ii), we have by (36) that $g(\lambda) = a\cdot\lambda + b$ for all real numbers $\lambda \in [0, 1]$. □

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