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The Heckscher–Ohlin–Samuelson Model and the Cambridge Capital Controversies*

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Abstract

This paper examines the validity of the factor price equalisation theorem (FPET) in relation to capital theory. Additionally, it presents a survey of the literature on Heckscher–Ohlin–Samuelson (HOS) models that treat capital as a primary factor, beginning with Samuelson (1953). Furthermore, this paper discusses the Cambridge capital controversy, which contends that marginal productivity theory does not hold when capital is assumed to be as a bundle of reproducible commodities instead of as a primary factor. Consequently, it is shown that under this assumption, the FPET does not hold, even when there is no reversal of capital intensity. This paper also demonstrates that the recent studies on the dynamic HOS trade theory generally ignore the difficulties posed by the capital controversies and are thereby able to conclude that the FPET holds even when capital is modelled as a reproducible factor. Our analysis suggests that there is a need for a basic theory of international trade that does not rely on factor price equalisation and a model that formulates capital as a bundle of reproducible commodities.

JEL Classification Code: B51, D33, F11.

Keywords: factor price equalisation, capital as the bundle of reproducible commodities, reswitching of techniques, capital reversing.

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1 Introduction

'Globalisation' is undoubtedly one of the key words that characterises the 21st Century. It would be defined sufficiently as the integration of markets for goods, services, capital, and labour (which had formerly been segmented by political barriers) into a single 'world market', although the various definitions of 'globalisation' are proposed as Wolf (2004) discusses.

The driving force behind modern globalisation was, at first, the establishment of a free trade system under the International Monetary Fund and the General Agreement on Tariffs and Trade (IMF-GATT), which gradually reduced tariffs after the end of the WWII. Thereafter, the collapse of the IMF's fixed exchange rate system and its replacement by the flexible exchange rate system further drove globalisation. Additionally, the relaxation and abolishment of various regulations, which were the product of the counter-Keynesian revolution, resulted in the free international movement of capital. Finally, the WTO was founded and it established comprehensive rules for international transactions of goods, finances, information (i.e. communication), intellectual property, and services.[1]

Almost all economic theories have been supportive of globalisation. Indeed, the classical economists of the 19th century, such as Smith, Ricardo, J.S. Mill, and Marx, asserted that open economies were superior to closed ones.[2] Moreover, nowadays, neo-classical economics also assert the superiority of an open economy by establishing the Heckscher–Ohlin–Samuelson (HOS) model. As a basis of supporting globalisation, economists typically refer to the potential gains from trade (Anderson, 2008) enjoyed by every economic agent under open economies. To explain the potential gains from trade, Ricardo's theory of 'comparative advantage' remains one of the cornerstones of international economics, while neo-classical economics also argue 'comparative advantage' within the HOS framework.

However, there are several differences between neo-classical and classical (including Marx) models on gains from trade; the former supposes that every country is faced with a common set of techniques, but differ in terms of factor endowments, while the latter assumes that each country is endowed with its own techniques which may vary from each other. Note that in the modern economy, globalisation actively promotes the international movement of *not only* goods, services, capital, and labour, *but also* of information and knowledge, which allows everyone to access common information and knowledge of production technology from anywhere, at least in the long run. In order to capture such a feature of the modern economy, it would be admissible to assume that every country is faced with a common set of techniques, which is formalized by a common production possibility set, as in the HOS model. It should be noted, however, that access to information and knowledge of production technology does not necessarily imply that every country can use them effectively. In order for a country to use a technique, it must have the necessary capital formation and labour force. Given the imperfection of international factor markets, the choice of a technique is dependent on the country's factor endowments.

In order to analyse globalisation with the HOS model, the validity of a set of theorems (i.e. the HO theorem, factor price equalisation theorem, the Stolper-Samuelson theorem, and the Rybczynski theorem) must be examined. Although the discovery of the Leontief Paradox (Leontief, 1953, 1956) precipitated such an examination, we focus on the factor price equalisation theorem (FPET), which is the cornerstone of the HOS model. According to this theorem, the equilibrium international price, as determined by free trade, ensures the equalisation of factor prices. Thus, it is important to determine whether or not factor prices tend to converge in modern globalisation.

In their analysis of the US current account imbalance, Obstfeld and Rogoff (2005) reveal that the income return on US-owned assets exceeded that on US liabilities by an average of 1.2% a year from 1983 to 2003. Furthermore, the return on US foreign investments, including capital gains, exceeded that on US liabilities by a remarkable 3.1% during the same period. If the return is regarded as a measurement of factor price, how can this persistent difference be explained? This paper seeks to answer this question by reviewing the development of the HOS model.[3]

[1] See Wolf (2004) concerning the history of the world economy's construction in detail.

[2] However, Malthus and List exceptionally criticise the free trade doctrine. It is well known that Malthus (1815) criticises the free trade system for its effects on food security and the stability of prices. List (1904) also criticises free trade for its failure to protect infant industries.

[3] While we have a special interest in capital, several studies examine the inexplicable relationship between international trade and wage disparities; see, for example, Kurokawa (2014).

We pay particular attention to the relationship between the theoretical development of the HOS model and the outcome of the Cambridge capital controversies, which revealed that the neo-classical principle of marginal productivity does not, in general, hold. The neo-classical production function treats capital as a primary factor of production, and thus, its amount is given independently of the price system. If capital is treated as a bundle of reproducible commodities, however, the neo-classical theory does not hold. Assuming the neo-classical production function, the rate of profit maintains a one-to-one correspondence with a technique. If capital consists of a bundle of reproducible commodities, however, a technique may correspond to some rate of profit. This phenomenon is termed *the reswitching of techniques*. Additionally, if capital consists of a bundle of reproducible commodities (unlike the principle of marginal productivity argues), then the monotonically decreasing relationship between capital intensity and the rate of profit does not generally hold. In other words, capital intensity may rise as the rate of profit increases, a phenomenon termed *capital reversing*. As we shall argue later, the outcome of the controversies may be used to re-examine the validity of the HOS model as it assumes the neo-classical production function and as capital is a primary factor. Neo-Ricardians who were influenced by Sraffa (1960), such as Steedman, Metcalfe, and Mainwaring, have thus far conducted the majority of such re-evaluations. By using the Leontief model with alternative techniques, they assert that if capital consists of heterogeneously reproducible commodities, then the FPET does not necessarily hold.

The neo-Ricardian arguments may provide a clear explanation for the persistent differences between the US' returns and those of the rest of the world. Since it is clear that in the globalised economy, capital is not a primary factor, but rather composed of a bundle of reproducible commodities, the FPET does not necessarily hold. Even if the globalised economy were perfectly competitive as the theory assumes, there could still be differences in the returns.

The paper is organised as follows: Section 2 presents a brief survey of the literature on the traditional HOS model, in which capital is treated as a primary factor. Section 3 deals with the neo-Ricardian critiques of the FPET and presents a numerical example of the two-integrated-sector model, in which the FPET does not hold despite a lack of sectoral capital intensity reversal. The model is based on the Leontief model with alternative techniques. It should be noted that our example is more rigorous than that presented by Metcalfe and Steedman (1972). Section 4 reviews neo-classical economists' counterarguments against the critiques popularized by the controversies. We show that Burmeister (1978), who presented the most influential form of the HOS model after the controversies, merely assumed that the inconvenient phenomena emphasized by the controversies would not occur. Section 5 presents our concluding remarks.

Throughout the paper, we assume that international trade does not incur any costs (e.g. transportation costs and tariffs) except for the direct cost of production; there is no perfect specialisation, and as such, every country produces all commodities; and that there is no joint production unless otherwise stated.

2 The HOS Model with Capital as a Primary Factor of Production

In this section, we examine the traditional HOS model in which capital is treated as a primary factor of production. Although Heckscher and Ohlin (1991) define the structure of comparative advantage as the difference in countries' factor endowments and put forward a prototype of the FPET, it is Samuelson (1953) who formalises it by using the general equilibrium theory.

Subsequently, Gale and Nikaido (1965) and Nikaido (1968) develope the Samuelsonian formulation; Samuelson (1966a) and Nikaido (1972) define capital intensity as the relative share of factor costs; and Mas-Collel (1979a,b) further developes Nikaido's (1972) formulation. Kuga (1972) characterises the FPET without using a Jacobian matrix for the cost function; Blackorby et al. (1993) extend Kuga (1972) by allowing for decreasing returns to scale and intermediate goods.

2.1 Samuelson (1953)

Following Samuelson (1948, 1949), wherein he proved the FPET with a two-country, two-commodity, two-factor model, Samuelson (1953) extends the theorem by using the general equilibrium model. By simplifying Samuelson's (1953) model, wherein there are n commodities and n primary factors, we can define the equilibrium condition as follows:

$$\mathbf{p} \leqq \mathbf{wA}(\mathbf{w}), \tag{1}$$
$$[\mathbf{p} - \mathbf{wA}(\mathbf{w})]\,\mathbf{X} = 0, \tag{2}$$
$$\mathbf{A}(\mathbf{w})\,\mathbf{X} = \mathbf{V}, \tag{3}$$

where $\mathbf{p} \equiv [p_i], \mathbf{w} \equiv [w_i], \mathbf{X} \equiv [X_i], \mathbf{V} \equiv [V_i] \in \mathbb{R}^n$ denote the vector of commodity prices, factor prices, output, and factor endowments. $\mathbf{A}(\mathbf{w})$ is the physical input coefficient matrix under which the unit cost is minimised given the primary factor price vector, $\mathbf{w}$; and thus, $\mathbf{wA}(\mathbf{w})$ denotes the unit cost function. (1) is the condition that allows for competitive equilibrium prices; (2) is the condition for the commodity market's equilibrium; and (3) is the condition which establishes the full utilisation of factors.

Let us assume the neo-classical production function, $X_j = f_j(V_{1j}, \cdots, V_{nj})$, where $\sum_{j=1}^{n} V_{ij} = V_i$.[4] As the function is homogeneous of degree one, it can be rewritten as follows:

$$1 = f_j(a_{1j}, \cdots, a_{nj}), \text{ where } a_{ij} \equiv \frac{V_{ij}}{X_j}. \tag{4}$$

It should be noted that the function satisfies the following assumptions:

$$\frac{w_i}{p_j} \geqq \frac{\partial f_j(a_{1j}, \cdots, a_{nj})}{\partial a_{ij}} \text{ for } i, j = 1, \cdots, n, \tag{5}$$

$$\frac{\partial f_j}{\partial a_{ij}} \geqq 0.$$

The production set satisfies the free disposal condition. For $\mathbf{V}'_j \equiv \left(V'_{1j}, \cdots, V'_{nj}\right), \mathbf{V}''_j \equiv \left(V''_{1j}, \cdots, V''_{nj}\right)$, and $\forall \lambda \in (0,1)$, moreover, the following is satisfied:

$$f_j\left(\lambda \mathbf{V}'_j + (1-\lambda)\mathbf{V}''_j\right) \geqq \lambda f_j\left(\mathbf{V}'_j\right) + (1-\lambda) f_j\left(\mathbf{V}''_j\right). \tag{6}$$

(6) indicates that f_j is a concave function. If all commodities are produced and all factors are utilised in every industry, then the equality holds in both (1) and (5).

Let us denote the unit cost function as follows: $c(\mathbf{w}) \equiv \mathbf{wA}(\mathbf{w}) = [c_j(\mathbf{w})]$, where $c_j(\mathbf{w}) \equiv \sum_{i=1}^{n} w_i a_{ij}(\mathbf{w})$. If the neo-classical production function is assumed, then function $c(\mathbf{w})$ has the following properties:[5]

Assumption 2.1.1: $c(\mathbf{w})$ is differentiable with respect to $\mathbf{w}$.

Assumption 2.1.2: $c(\mathbf{w})$ is a homogeneous function of degree one.

Assumption 2.1.3: $c(\mathbf{w})$ is concave with respect to $\mathbf{w}$.

Assumption 2.1.4: $c(\mathbf{w})$ is monotonically increasing with respect to $\mathbf{w}$.

The FPET holds that the cost function, c: $\mathbf{w} \mapsto \mathbf{p}$, is **global univalent**. Let us consider a simple case wherein $n = 2$. In this case, the FPET's assumption of no factor intensity reversal causes the factor prices to equalise in free trade equilibrium. The factor intensity of industry 1 is given by $a_{11}(\mathbf{w})/a_{21}(\mathbf{w})$ and that of industry 2 is $a_{12}(\mathbf{w})/a_{22}(\mathbf{w})$. An absence of factor intensity reversal means that

$$\begin{cases} \forall \mathbf{w} \geq \mathbf{0},\ a_{11}(\mathbf{w})\,a_{22}(\mathbf{w}) - a_{12}(\mathbf{w})\,a_{21}(\mathbf{w}) > 0 \text{ or} \\ \forall \mathbf{w} \geq \mathbf{0},\ a_{11}(\mathbf{w})\,a_{22}(\mathbf{w}) - a_{12}(\mathbf{w})\,a_{21}(\mathbf{w}) < 0. \end{cases} \tag{7}$$

[4] See Burmeister and Dobell (1970, pp. 8–12) with respect to the neo-classical production function in detail.

[5] See, for example, Mas-Colell et al. (1995, p. 141).

If (7) is satisfied, then the FPET holds. This is because $\mathbf{A}(\mathbf{w})$ is non-singular and thus has an inverse matrix. Therefore, $\mathbf{w} = \mathbf{pA}(\mathbf{w})^{-1}$. In this case, it is shown that $c(\mathbf{w})$ is bijective over the range of the cost function, $\mathcal{P} \equiv \{\mathbf{p} \in \mathbb{R}_+^n \mid \exists \mathbf{w} \in \mathbb{R}_+^n : c(\mathbf{w}) = \mathbf{p}\}$. This directly implies that c is global univalent.[6]

If we allow factor intensity reversal to occur (i.e. (7) is not satisfied), due to the continuity of function $c(\mathbf{w})$, $a_{11}(\mathbf{w}') a_{22}(\mathbf{w}') - a_{12}(\mathbf{w}') a_{21}(\mathbf{w}') = 0$ holds for $\mathbf{A}(\mathbf{w}')$, which is chosen for $\mathbf{w}'$. If the final commodity prices in the incompletely specialised-trade equilibrium are given by $\mathbf{p}' = \mathbf{w}'\mathbf{A}(\mathbf{w}')$, then $\mathbf{A}(\mathbf{w}')$ is singular and does not have an inverse matrix. Therefore, there are an infinite number of factor price vectors that would satisfy the equation; in other words, the factor prices do not equalise.

Partially differentiating $c_j(\mathbf{w})$ with respect to w_i yields: $\frac{\partial p_j}{\partial w_i} = \frac{\partial c_j(\mathbf{w})}{\partial w_i} = a_{ij}(\mathbf{w}) + \sum_{h=1}^{2} w_h \frac{\partial a_{hj}(\mathbf{w})}{\partial w_i}$. Given that $\sum_{h=1}^{2} \frac{\partial f_j}{\partial a_{hj}} \frac{\partial a_{hj}}{\partial w_i} = 0$ is obtained by differentiating (4) and $\frac{\partial f_j}{\partial a_{hj}} = \frac{w_h}{p_j}$ is obtained from (5), $\frac{1}{p_j} \sum_{h=1}^{2} w_h \frac{\partial a_{hj}(\mathbf{w})}{\partial w_i} = 0$ is obtained. As $p_j > 0$, we find that:

$$\frac{\partial p_j}{\partial w_i} = a_{ij}(\mathbf{w}), \quad i, j = 1, 2. \tag{8}$$

This implies that partially differentiating the price equation yields the input coefficient, a_{ij}. Samuelson (1953) shows that the cost function's non-vanishing Jacobian matrix is the sufficient condition for the validity of FPET in the case of $n = 2$:

$$\det \begin{bmatrix} a_{11}(\mathbf{w}) & a_{12}(\mathbf{w}) \\ a_{21}(\mathbf{w}) & a_{22}(\mathbf{w}) \end{bmatrix} = \det \begin{bmatrix} \frac{\partial c_1(\mathbf{w})}{\partial w_1} & \frac{\partial c_2(\mathbf{w})}{\partial w_1} \\ \frac{\partial c_1(\mathbf{w})}{\partial w_2} & \frac{\partial c_2(\mathbf{w})}{\partial w_2} \end{bmatrix} \neq 0.$$

Furthermore, Samuelson (1953) extends the case of $n = 2$ to a more general scenario of $n \geqq 3$ and conjectures the sufficient condition of the validity of FPET as follows:

$$\frac{\partial c_1(\mathbf{w})}{\partial w_1} \neq 0, \det \begin{bmatrix} \frac{\partial c_1(\mathbf{w})}{\partial w_1} & \frac{\partial c_2(\mathbf{w})}{\partial w_1} \\ \frac{\partial c_1(\mathbf{w})}{\partial w_2} & \frac{\partial c_2(\mathbf{w})}{\partial w_2} \end{bmatrix} \neq 0, \cdots, \det \begin{bmatrix} \frac{\partial c_1(\mathbf{w})}{\partial w_1} & \cdots & \frac{\partial c_n(\mathbf{w})}{\partial w_1} \\ \vdots & \ddots & \vdots \\ \frac{\partial c_1(\mathbf{w})}{\partial w_n} & \cdots & \frac{\partial c_n(\mathbf{w})}{\partial w_n} \end{bmatrix} \neq 0. \tag{9}$$

(9) indicates that the sufficient condition for the validity of the FPET is that the successive principal minors of the cost function's Jacobian matrix be non-vanishing.

Furthermore, it is clear that the condition is also valid for the Leontief production function. This is because (8) may simply be rewritten as $\frac{\partial p_j}{\partial w_i} = a_{ij}$, which has constant coefficients. Therefore, the Jacobian matrix has one-signed principal minors even when $\mathbf{w}$ changes.

Samuelson's (1953) use of the cost function's Jacobian matrix to characterise the condition for the validity of the FPET had a decisive impact on the direction of later research.[7]

2.2 The Application of Jacobian Matrix

Gale and Nikaido (1965) and Nikaido (1968) point out a major flaw in Samuelson (1953).[8] Before proceeding with our analysis, let us first define the following matrices:

Definition 2.2.1: A square matrix, $\mathbf{A}$, is termed a P-matrix if all the principal minors are positive.

[6] See the Appendix for the rigorous proof in the case of $n = 2$.

[7] See Chipman (1966) for a description of other HOS models based on the general equilibrium theory.

[8] Gale and Nikaido (1965) provide a counter-example for condition (9). They suppose the mapping $F(\mathbf{x}) \equiv [f_i(\mathbf{x})]$ as defined below:

$$\begin{cases} f_1(x_1, x_2) = e^{2x_1} - x_2^2 + 3, \\ f_2(x_1, x_2) = 4e^{2x_1} x_2 - x_2^3. \end{cases}$$

The successive principal minors can then be given as:

Definition 2.2.2: A square matrix, $\mathbf{A}$, is termed an N-matrix if all the principal minors are negative. An N-matrix can be further divided into two categories:
i) An N-matrix is said to be of the *first category* if $\mathbf{A}$ has at least one positive element.
ii) An N-matrix is said to be of the *second category* if all of the elements are non-positive.

Let the mapping $f : \Omega \to \mathbb{R}^n$ satisfy the following assumptions:

Assumption 2.2.1: Ω is a closed rectangular region in $\mathbb{R}^n$.[9]

Assumption 2.2.2: Given that the mapping $f(\mathbf{x}) \equiv [f_j(\mathbf{x})]$ $(\mathbf{x} \in \Omega, j = 1, 2, \cdots, n)$, $f_j(\mathbf{x})$ is monotonically increasing and totally differentiable on Ω:

$$\mathrm{d}f_j(\mathbf{x}) = \sum_{j=1}^{n} \frac{\partial f_j(\mathbf{x})}{\partial x_i} \mathrm{d}x_i, \ (j = 1, 2, \cdots, n).$$

Consequently, the following theorem holds:

Theorem 1 (Gale and Nikaido, 1965; Inada, 1971; Nikaido, 1968): For a given vector, $\mathbf{p} \equiv [p_j]$, mapping $\mathbf{p} = f(\mathbf{x})$ is global univalent if either (a) or (b) holds:

(a) The Jacobian matrix of $f(\mathbf{x})$, is everywhere a P-matrix in Ω.
(b) The Jacobian matrix is continuous and is everywhere an N-matrix in Ω.

Proof: See Nikaido (1968, pp. 370–371). ■

As Ethier (1984, p. 151) points out, Gale and Nikaido's (1965) assumptions regarding the mapping, f, are quite general and their conditions for global univalence are purely mathematical. Therefore, it is necessary to clarify the kinds of assumptions that shall be imposed on the cost function in order to capture a standard economic environment.

Samuelson (1966a) conjectures that 'factor intensity' could be defined by the share of the increase in the cost of factor i relative to the increase in the cost of production per unit; in other words, for price equation, $\mathbf{p} = c(\mathbf{w})$, the factor intensity, α_{ij}, is given as:

$$\alpha_{ij} \equiv \frac{c_{ij}(\mathbf{w})\, w_i}{p_j}, \ (\forall i, j = 1, \ldots, n),$$

where $c_{ij}(\mathbf{w}) \equiv \frac{\partial c_j(\mathbf{w})}{\partial w_i}$. α_{ij} is the share of the rate of increase in the cost of factor i relative to that in the cost of producing one unit of commodity j. Let us define matrix $\widetilde{\mathbf{A}} \equiv [\alpha_{ij}]$ $(i, j = 1, 2, \cdots, n)$.[10] Moreover, let us assume that $\widetilde{\mathbf{A}}$ has successive principal minors whose absolute values are bounded from below by constant, positive numbers, δ_k $(k = 1, 2 \cdots, n)$, if its rows and columns are adequately renumbered:[11]

$$\frac{\partial f_1}{\partial x_1} = 2e^{2x_1} > 0,$$
$$\begin{vmatrix} \partial f_1/\partial x_1 & \partial f_1/\partial x_2 \\ \partial f_2/\partial x_1 & \partial f_2/\partial x_2 \end{vmatrix} = \begin{vmatrix} 2e^{2x_1} & -2x_2 \\ 8e^{2x_1}x_2 & 4e^{2x_1} - 3x_2^2 \end{vmatrix} = 2e^{2x_1}\left(4e^{2x_1} + 5x_2^2\right) > 0,$$

holds for $\forall \mathbf{x}$; therefore, (9) is satisfied. However, $F(0, 2) = F(0, -2) = (0, 0)$, which precludes global univalence.

[9] A closed rectangular region is defined as follows:

$$\Omega \equiv \{\mathbf{x} |\, p_i \leqq x_i \leqq q_i, i = 1, 2, \cdots, n\},$$

where $-\infty < p_i < q_i < +\infty$.

[10] $\alpha_{ij} \geqq 0$ and $\sum_{i=1}^{n} \alpha_{ij} = 1$ are obtained using the Euler Theorem and the homogeneity of the cost functions. Therefore, $\widetilde{\mathbf{A}}$ is a stochastic matrix.

[11] (10) is equivalent to (7) if there are two-commodities and two-factors.

$$\left|\det\begin{bmatrix}\alpha_{11} & \cdots & \alpha_{1k}\\ \vdots & \ddots & \vdots\\ \alpha_{k1} & \cdots & \alpha_{kk}\end{bmatrix}\right| \geqq \delta_k,\ k=1,2\cdots,n. \tag{10}$$

Consequently, the following theorem holds:

Theorem 2 (Nikaido, 1972): If $c_j(\mathbf{w})$ satisfies Assumption 2.1.1~2.1.4, then the price equation, $\mathbf{p}=c(\mathbf{w})$, is completely invertible for the given $\mathbf{p}>\mathbf{0}$.[12]

Proof: See appendix. ■

Theorem 2 verifies Samuelson's conjecture.[13]

Stolper and Samuelson (1941) also investigated the relationship between final commodity prices and factor prices; thus, the approach of using the cost function's Jacobian matrix is applied to the generalisation of the Stolper-Samuelson Theorem. Chipman (1969) proposed the following two criteria:

i) Weak Stolper–Samuelson Criterion (WSS): An increase in p_j leads to a more than proportional increase in the price of the corresponding factor w_j. The price of factor $w_i\ (i\neq j)$ may increase, but the rate of increase is smaller than that of w_j:

$$\frac{\partial\ln w_j}{\partial\ln p_j}>1,\ \frac{\partial\ln w_j}{\partial\ln p_j}>\frac{\partial\ln w_i}{\partial\ln p_j}.$$

ii) Strong Stolper–Samuelson Criterion (SSS): An increase in p_j decreases all factor prices except for that of w_j:

$$\frac{\partial\ln w_i}{\partial\ln p_j}<0,\ \text{if } i\neq j.$$

In order to satisfy the WSS condition, the inverse of $\widetilde{\mathbf{A}}$ must exist and its diagonal elements must be greater than 1 and its non-diagonal elements.[14] In other words, letting $\widetilde{\mathbf{A}}^{-1}\equiv\left[\alpha^{ij}\right]\ (i,j=1,2,\cdots,n)$, the WSS condition implies $\alpha^{jj}>1$ and $\alpha^{jj}>\alpha^{ij}\ (i\neq j)$. Similarly, the SSS condition, in terms of $\widetilde{\mathbf{A}}^{-1}$, implies that $\alpha^{ij}<0\ (i\neq j)$. While Chipman (1969) proves the case of $n\leqq 3$, Uekawa (1971) and Uekawa et al. (1972) rigorously prove the condition for the validity of the Stolper–Samuelson theorem in the case of $n\geqq 4$.[15]

[12] 'Completely invertibility' means that $\mathbf{p}=c(\mathbf{w})\neq c(\mathbf{w}')=\mathbf{p}'$ for arbitrarily positive vectors $\mathbf{w}\neq\mathbf{w}'$ and a unique $\mathbf{w}>\mathbf{0}$ exists such that $\mathbf{p}=c(\mathbf{w})$ for $\forall\mathbf{p}>\mathbf{0}$.

[13] Samuelson's own summary can be found in Samuelson (1967). Moreover, Stiglitz (1970) constructs a dynamic HOS model by introducing a relationship between savings and investment, and he derives the condition for the FPET. See Smith (1984) with respect to the dynamic HOS model in detail.

[14] Chipman (1969) discusses the relationship between Gale and Nikaido's (1965) condition for the FPET and the WSS condition. When $n=2$, the condition is equivalent to the WSS condition, but is not if $n\geqq 3$. Suppose that $\mathbf{w}^0$ is determined for a given $\mathbf{p}$ and $\pi^0=\varphi\left(\omega^0\right)$ (which is defined in the proof of Theorem 2). Let us suppose that its Jacobian matrix is given as follows:

$$\varphi'\left(\boldsymbol{\omega}^0\right)=\begin{bmatrix}0.55 & 0.40 & 0.05\\ 0.05 & 0.50 & 0.45\\ 0.25 & 0.35 & 0.40\end{bmatrix}.$$

$\varphi'\left(\boldsymbol{\omega}^0\right)$ is a stochastic matrix. Furthermore, $\varphi\left(\boldsymbol{\omega}\right)$ is the differentiable and monotonically increasing function, and all principal minors of $\varphi'\left(\boldsymbol{\omega}^0\right)$ are positive, namely $\varphi'\left(\boldsymbol{\omega}^0\right)$ is a P-matrix and satisfies Theorem 1. This means that the FPET holds. However, we obtain:

$$\left[\varphi'\left(\boldsymbol{\omega}^0\right)\right]^{-1}=\begin{bmatrix}0.77 & -2.59 & 2.82\\ 1.68 & 3.77 & -4.45\\ -1.95 & -1.68 & 4.64\end{bmatrix}.$$

This means that the WSS condition does not hold.

[15] The Stolper–Samuelson theorem is similarly generalised by Inada (1971), Kemp and Wegge (1969), Morishima (1976), and Wegge and Kemp (1969). See Ethier (1984) for further research on the theorem.

2.3 Kuga (1972)

The preceding analyses can only be applied to cases wherein the number of final commodities is equal to that of factors. In order to overcome this limitation, Kuga (1972) uses a new approach to characterise the condition for the FPET, which he terms the 'differentiation method'.

Let us assume that the general production possibility frontier is given as follows:

$$X_1 = T(\mathbf{V}; \mathbf{X}),$$

where X_1 denotes the output of commodity 1, $\mathbf{V} \in \mathbb{R}_+^r$ is the factor endowment, and $\mathbf{X} \in \mathbb{R}_+^{n-1}$ is the output vector of commodity 2 to n. Moreover, T satisfies the following assumptions:

Assumption 2.3.1: T is positively homogeneous of degree one with respect to $(\mathbf{V}; \mathbf{X})$.

Assumption 2.3.2: T is concave with respect to $(\mathbf{V}; \mathbf{X})$.

Assumption 2.3.3: T is strictly concave with respect to $\mathbf{X}$ for any fixed $\mathbf{V}$.

Assumption 2.3.4: T is twice differentiable with respect to $(\mathbf{V}; \mathbf{X})$.

Let the price of commodity 1 be the numéraire; then, the problem is expressed as follows:

$$\max T(\mathbf{V}; \mathbf{X}) + \sum_{j=2}^{n} p_j X_j, \tag{11}$$

the solution of which is given by:

$$p_j = -\frac{\partial T(\mathbf{V}; \mathbf{X})}{\partial X_j}, \quad j = 2, 3, \cdots, n, \tag{12}$$

Thanks to the Berge maximum theorem, we can see that the set of solutions to (11), X_j, is upper hemi-continuous with respect to $\mathbf{V}$ for a given $\mathbf{p}$. Moreover, because of Assumption 2.3.3, the set is singleton. Therefore, the solution, X_j, is the continuous single valued function of $\mathbf{V}$:

$$\mathbf{X} = X(\mathbf{V}; \mathbf{p}).$$

The price of factor i is given by:

$$w_i = \frac{\partial T(\mathbf{V}; X(\mathbf{V}; \mathbf{p}))}{\partial V_i}, \quad i = 1, 2, \cdots, r. \tag{13}$$

The equalisation of factor prices in this model implies that factor price w_i is solely dependent on the commodity price that is determined by free trade, and thus the right-hand side of (13) is kept constant with respect to the variation of $\mathbf{V}$.

By partially differentiating (13) with respect to V_τ $(\tau = 1, 2, \cdots, r)$, we obtain:

$$\frac{\partial w_i}{\partial V_\tau} = \frac{\partial^2 T}{\partial V_\tau \partial V_i} + \sum_{j=2}^{n} \frac{\partial^2 T}{\partial X_j \partial V_i} \frac{\partial X_j}{\partial V_\tau}, \quad \tau = 1, 2, \cdots, r, \tag{14}$$

in matrix form this is written as

$$\mathbf{w}_V = \mathbf{M}_1 + \mathbf{M}_2 \mathbf{X}_V, \tag{15}$$

where $\mathbf{w}_V \equiv \begin{bmatrix} \frac{\partial w_1}{\partial V_1} & \cdots & \frac{\partial w_r}{\partial V_1} \\ \vdots & \ddots & \vdots \\ \frac{\partial w_1}{\partial V_r} & \cdots & \frac{\partial w_r}{\partial V_r} \end{bmatrix}$, $\mathbf{M}_1 \equiv \begin{bmatrix} \frac{\partial^2 T}{\partial V_1^2} & \cdots & \frac{\partial^2 T}{\partial V_1 \partial V_r} \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 T}{\partial V_r \partial V_1} & \cdots & \frac{\partial^2 T}{\partial V_r^2} \end{bmatrix}$,

$$\mathbf{M}_2 \equiv \begin{bmatrix} \frac{\partial^2 T}{\partial V_1 \partial X_2} & \cdots & \frac{\partial^2 T}{\partial V_1 \partial X_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 T}{\partial V_r \partial X_2} & \cdots & \frac{\partial^2 T}{\partial V_r \partial X_n} \end{bmatrix}, \mathbf{X}_V \equiv \begin{bmatrix} \frac{\partial X_2}{\partial V_1} & \cdots & \frac{\partial X_2}{\partial V_r} \\ \vdots & \ddots & \vdots \\ \frac{\partial X_n}{\partial V_1} & \cdots & \frac{\partial X_n}{\partial V_r} \end{bmatrix}.$$

Similarly, partially differentiating (12) with respect to V_τ yields:

$$\frac{\partial^2 T}{\partial X_j \partial V_\tau} + \sum_{l=2}^{n} \frac{\partial^2 T}{\partial X_l \partial X_j} \frac{\partial X_l}{\partial V_\tau} = 0, \quad j = 2, 3, \cdots, n, \tau = 1, 2, \cdots, r. \tag{16}$$

In matrix from, this is written as $\mathbf{M}_2^T + \mathbf{M}_3 \mathbf{X}_V = \mathbf{0}$, where $\mathbf{M}_3 \equiv \begin{bmatrix} \frac{\partial^2 T}{\partial X_2^2} & \cdots & \frac{\partial^2 T}{\partial X_2 \partial X_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 T}{\partial X_n \partial X_2} & \cdots & \frac{\partial^2 T}{\partial X_n^2} \end{bmatrix}$ and the superscript T denotes the transpose. Thanks to Assumption 2.3.3, the Hessian matrix $\mathbf{M}_3$ has an inverse such that $\mathbf{X}_V = -\mathbf{M}_3^{-1} \mathbf{M}_2^T$ holds.[16] When this is combined with (15), we obtain:

$$\mathbf{w}_V = \mathbf{M}_1 - \mathbf{M}_2 \mathbf{M}_3^{-1} \mathbf{M}_2^T. \tag{17}$$

In order for the FPET to hold, $\mathbf{w}_V = \mathbf{0}$ must hold:

$$\mathbf{M}_1 = \mathbf{M}_2 \mathbf{M}_3^{-1} \mathbf{M}_2^T \tag{18}$$

(17) and (18) have economic implications. Partially differentiating (12) with respect to p_j yields $1 = -\sum_{k=2}^{n} \frac{\partial^2 T}{\partial X_k \partial X_j} \frac{\partial X_k}{\partial p_j}$, which can be rewritten in matrix form as:

$$\mathbf{I} = -\mathbf{M}_3 \mathbf{X}_p, \tag{19}$$

where $\mathbf{I}$ is an identity matrix of order $n-1$ and $\mathbf{X}_p \equiv \begin{bmatrix} \frac{\partial X_2}{\partial p_2} & \cdots & \frac{\partial X_2}{\partial p_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial X_n}{\partial p_2} & \cdots & \frac{\partial X_n}{\partial p_n} \end{bmatrix}$. Similarly, partially differentiating (13) with respect to p_j yields

$$\frac{\partial w_i}{\partial p_j} = \sum_{l=2}^{n} \frac{\partial^2 T}{\partial X_l \partial V_i} \frac{\partial X_l}{\partial p_j} \left(i = 1, 2, \cdots, r, j = 1, 2, \cdots, n\right),$$

which can be written in matrix form as:

$$\mathbf{w}_p = \mathbf{M}_2 \mathbf{X}_p, \tag{20}$$

where $\mathbf{w}_p \equiv \begin{bmatrix} \frac{\partial w_1}{\partial p_2} & \cdots & \frac{\partial w_1}{\partial p_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial w_r}{\partial p_2} & \cdots & \frac{\partial w_r}{\partial p_n} \end{bmatrix}$. Consequently, $\mathbf{M}_3^{-1} = -\mathbf{X}_p$ holds from (19), as does $\mathbf{M}_2 = \mathbf{w}_p \mathbf{X}_p^{-1}$ from (20). Because of (17), we therefore obtain:

$$\mathbf{w}_V = \mathbf{M}_1 + \mathbf{w}_p \mathbf{M}_2^T. \tag{21}$$

The variations in factor endowments, V_τ, tend to give rise to the variations in factor prices and output. The elements of $\mathbf{M}_1$, $\frac{\partial^2 T}{\partial V_i \partial V_\tau}$ $(i = 1, \cdots, r)$, indicate the variation of factor prices vary in response to variation in V_τ when there is no adjustment in X_j by an amount of $\frac{\partial X_j}{\partial V_\tau}$ $(j = 1, \cdots, n)$ (i.e. $\frac{\partial X_j}{\partial V_\tau} = 0$). The elements

[16] Assumption 2.3.3 says that T is strictly concave with respect to $\mathbf{X}$. This implies that the Hessian matrix of T is negative definite. A square matrix is negative definite if and only if its inverse is negative definite (Mas-Colell et al., 1995, p. 936); therefore, $\mathbf{M}_3$ is invertible.

of $\mathbf{M}_2$, $\frac{\partial^2 T}{\partial V_\tau \partial X_j}$, indicate the discrepancies between international prices, p_j, and domestic commodity production prices, $\frac{\partial T}{\partial X_j}$, in response to variations in V_τ when there is no adjustment in $\frac{\partial X_j}{\partial V_\tau}$ (i.e. $\frac{\partial X_j}{\partial V_\tau} = 0$). On the contrary, the elements of $\mathbf{w}_p$ convey the adjustment in the w_i's through the adjustments in the X_j's corresponding to the marginal discrepancies in international prices. Therefore, $\mathbf{w}_p \mathbf{M}_2^T$ in (21) can be interpreted as the potential amount of adjustment in the w_i's through the X_j's corresponding to the discrepancies in $\mathbf{M}_2^T$. Consequently, $\mathbf{M}_1$ and $\mathbf{w}_p \mathbf{M}_2^T$ shall be termed the 'direct effect' and the 'adjustment effect', respectively. In order for the FPET to hold in this model ($\mathbf{w}_V = \mathbf{0}$), the direct effect must be just offset by the adjustment effect. By summarising the above analysis, we obtain:

Theorem 3: Under Assumption 2.3.1~2.3.4, the FPET holds if and only if the direct effect is offset by the adjustment effect.

Kuga (1972) assures the validity of the FPET by keeping the factor price independent of the factor endowment, which was an entirely different approach than previous models had used.

2.4 Mas-Colell (1979a, b)

Mas-Colell (1979a, b) uses the relative share matrix, $\widetilde{\mathbf{A}}$, rather than the Jacobian matrix of the cost function to characterise the condition for the FPET. In doing so, he makes the following assumptions regarding the cost function:

Assumption 2.4.1: $c(\mathbf{w})$ is a continuously differentiable function and homogeneous of degree one.

Assumption 2.4.2: $c : \mathbb{R}^n_{++} \to \mathbb{R}^n_{++}$

Although the cost function is usually assumed to be concave with respect to $\mathbf{w}$, only homogeneity is assumed here. Because of Assumption 2.4.2, the iso-cost curve is unbounded.

As the definition of the WSS condition shows, the relative cost share is related to the cost function in the following manner:

$$\alpha_{ij} \equiv \frac{w_i}{c_j(\mathbf{w})} \frac{\partial c_j(\mathbf{w})}{\partial w_i}.$$

Consequently, the following theorem holds.

Theorem 4 (Mas-Colell, 1979a): Under Assumptions 2.4.1~2.4.2, if, for some $\varepsilon > 0$, $\left|\det \widetilde{\mathbf{A}}\right| > \varepsilon$ holds for all $\mathbf{w} \in \mathbb{R}^n_{++}$, then $c(\mathbf{w})$ is a homeomorphism.

Proof: See the Appendix. ■

In other words, Theorem 4 implies that for all $\mathbf{p} \in \mathbb{R}^n_{++}$ the equation $\mathbf{p} = c(\mathbf{w})$ has a unique solution that continuously depends on $\mathbf{p}$.

Moreover, Mas-Colell (1979a) presented the condition that allows the cost function to be a homeomorphism when it is bounded (i.e. $c_j : \mathbb{R}^n_{+} \to \mathbb{R}^n_{+}$) by utilizing the relative share matrix.

Mas-Colell's assumptions regarding the cost function are generalisations of Nikaido (1972); the difference lies in the fact that the space of the commodity prices and that of factors are homeomorphisms. The linear homogeneity and concavity of the cost function allow Nikaido (1972) to claim complete invertibility, but complete invertibility does not require that the invertible mapping be continuous. Incidentally, global univalence does not require this continuity either.

2.5 Blackorby et al. (1993)

Blackorby et al. (1993) also characterise the necessary and sufficient conditions for the factor price equlisation, which is a generalisation of Kuga (1982) in that they allow the possibility of joint production and decreasing returns to scale. Note that, as Kuga (1972) does, Blackorby et al. (1993) also argue that the factor price equlisation must be solely dependent on commodity prices and, therefore, independent of factor endowments in all countries.

Suppose that the international economy consists of C countries, indexed as $c = 1, \cdots, C$. Moreover, there are M final commodities which are traded freely in international markets and N primary factors that each country is endowed with. Let $\mathbf{X}^c \in \mathbb{R}^M$ denote the production vector of country c, the positive elements of which represent the outputs and the negative elements of which represent the inputs. $\mathbf{V}^c \in \mathbb{R}^N$ denotes factor endowments of country c. Given the transformation function $T^c : \mathbb{R}^{M+N} \to \mathbb{R}$, a net output and factor endowment vector, $(\mathbf{X}^c, \mathbf{V}^c)$, is feasible if and only if $T^c(\mathbf{X}^c, \mathbf{V}^c) \leqq 0$. Furthermore, it should be noted that T^c satisfies the following assumptions:

Assumption 2.5.1: (i) $D^c \equiv \left\{(\mathbf{X}^c, \mathbf{V}^c) \in \mathbb{R}^{M+N} \middle| T^c(\mathbf{X}^c, \mathbf{V}^c) \leqq 0\right\}$ is a non-empty and closed convex set, and $(0, 0) \in D^c$; (ii) T^c is increasing in $\mathbf{X}^c \in \mathbb{R}^M$ and decreasing in $\mathbf{V}^c \in \mathbb{R}^N$; (iii) T^c is convex in $(\mathbf{X}^c, \mathbf{V}^c) \in \mathbb{R}^{M+N}$.

Assumption 2.5.2: T^c is continuous and twice differentiable.

The transformation function is related to the production function, G^c, as follows:

$$T^c(\mathbf{X}^c, \mathbf{V}^c) = 0 \Longleftrightarrow \mathbf{Y}^c = G^c(\mathbf{Z}^c, \mathbf{V}^c),$$

where $\mathbf{X}^c \equiv (\mathbf{Y}^c, \mathbf{Z}^c)$. $\mathbf{Y}^c$ and $\mathbf{Z}^c$ denote the net output and net input, respectively. Due to Assumption 2.5.1, G^c is concave and decreasing in $\mathbf{Z}^c$ as well as increasing in $\mathbf{V}^c$. Therefore, the two expressions shown below represent the same profit maximisation problem:

$$R^c(\mathbf{p}, \mathbf{V}^c) = \max_{\mathbf{X}^c} \{\mathbf{p}\mathbf{X}^c | T^c(\mathbf{X}^c, \mathbf{V}^c) \leqq 0\}, \tag{22}$$

$$R^c(\mathbf{p}, \mathbf{V}^c) = \max_{\mathbf{Y}^c, \mathbf{Z}^c} \{\mathbf{p}^y\mathbf{Y}^c + \mathbf{p}^z\mathbf{Z}^c | \mathbf{Y}^c \leqq G^c(\mathbf{Z}^c, \mathbf{V}^c)\}, \tag{23}$$

where $\mathbf{p} \equiv (\mathbf{p}^y, \mathbf{p}^z)$ is the price vector. $R^c(\mathbf{p}, \mathbf{V}^c)$ satisfies the same properties that the profit function generally does; that is, $R^c(\mathbf{p}, \mathbf{V}^c)$ is a homogeneous function of degree one, non-decreasing convex, and increasing concave in $\mathbf{V}^c$.

An equilibrium factor price vector $\mathbf{W}^c \in \mathbb{R}^N$ and a corresponding equilibrium production vector $\mathbf{X}^{c*} \in \mathbb{R}^M$ of country $c = 1, \cdots, C$, are respectively defined as follows:

$$\mathbf{W}^c \begin{cases} = \nabla_{\mathbf{V}} R^c(\mathbf{p}, \mathbf{V}^c), \text{ if } R^c \text{ is differentiable,} \\ \in \partial_{\mathbf{V}} R^c(\mathbf{p}, \mathbf{V}^c), \text{ if } R^c \text{ is not differentiable,} \end{cases} \tag{24}$$

$$\mathbf{X}^{c*} \begin{cases} = \nabla_{\mathbf{p}} R^c(\mathbf{p}, \mathbf{V}^c), \text{ if } R^c \text{ is differentiable,} \\ \in \partial_{\mathbf{p}} R^c(\mathbf{p}, \mathbf{V}^c), \text{ if } R^c \text{ is not differentiable,} \end{cases} \tag{25}$$

where $\partial_i R^c(\mathbf{p}, \mathbf{V}^c)$ denotes the sub-gradient set at $(\mathbf{p}, \mathbf{V}^c)$, where $i = \mathbf{p}, \mathbf{V}^c$.

Here, the equalisation of factor prices is defined as follows:

Definition 2.5.1 (factor price equalisation (FPE)): Equilibrium factor prices are equalised for countries $c = 1, \cdots, C$ if and only if there exists a non-empty, open, convex subset of commodity prices $\Pi \subseteq \mathbb{R}^M_+$, and for each $\mathbf{p} \in \Pi$, there exists a profile of non-empty, open, convex subsets of factor endowments, $(\Gamma^c(\mathbf{p}))_{c=1,\cdots,C}$ such that for each $\mathbf{p} \in \Pi$ there exists a vector $\mathbf{W} \in \mathbb{R}^N_+$ such that $\mathbf{W} = \nabla_{\mathbf{V}} R^c(\mathbf{p}, \mathbf{V}^c)$ for each country $c = 1, \cdots, C$ and for an arbitrary profile of factor endowments, $(\mathbf{V}^c)_{c=1,\cdots,C} \in \underset{c=1,\cdots,C}{\times} \Gamma^c(\mathbf{p})$.

As a preliminary step, let us introduce the following two concepts which play important roles in this model:

Definition 2.5.2 (Linear Segment): A vector $(\psi^c, \delta^c) = \left(\psi_y^c, \psi_z^c, \delta^c\right)$ is a linear segment of G^c at $(\mathbf{Z}^c, \mathbf{V}^c)$ if and only if there exists an $\varepsilon > 0$ such that

$$G^c\left(\mathbf{Z}^c, \mathbf{V}^c\right) + \lambda\psi_y^c = G^c\left(\mathbf{Z}^c + \lambda\psi_z^c, \mathbf{V}^c + \lambda\delta^c\right),$$

for all $\lambda \in (-\varepsilon, \varepsilon)$.

Definition 2.5.3 (Direction of linearity): A vector $(\psi^c, \delta^c) \in \mathbb{R}^{M+N}$ is a direction of linearity of T^c at $(\mathbf{X}^c, \mathbf{V}^c)$ if and only if there exists an $\varepsilon > 0$ such that

$$T^c\left(\mathbf{X}^c + \lambda\psi^c, \mathbf{V}^c + \lambda\delta^c\right) = 0,$$

for all $\lambda \in (-\varepsilon, \varepsilon)$.

While changes in factor endowments generally produce changes in the production vector, a direction of linearity means that a change in the feasible and efficient production vector, $\mathbf{X}^c$, precipitated by a change in factor endowments, δ^c, is linear; in other words, $\partial T^c\left(\mathbf{X}^c + \lambda\psi^c, \mathbf{V}^c + \lambda\delta^c\right) = \partial T^c\left(\mathbf{X}^c, \mathbf{V}^c\right)$. Therefore, a change in the production vector along a direction of linearity does not change the gradient vector of T^c.

According to Definitions 2.5.2 and 2.5.3, it is clear that (ψ^c, δ^c) is a direction of linearity of T^c if and only if (ψ^c, δ^c) is a linear segment of G^c. In what follows, for $c = 1, \cdots, C$, Π and $\Gamma^c(\mathbf{p})$ are of full dimension, and as such, the notation $\Gamma_N^c(\mathbf{p})$ is used to emphasise this. Here, the necessary and sufficient condition for the FPE to hold is given by the following theorem.

Theorem 5: Under Assumption 2.5.1, the FPE holds for each $\mathbf{p} \in \Pi$ and each $(\mathbf{V}^c)_{c=1,\cdots,C} \in \underset{c=1,\cdots,C}{\times} \Gamma_N^c(\mathbf{p})$ if and only if the following conditions hold:
1) there exist N vectors, $(\psi_i(\mathbf{p}), \delta_i(\mathbf{p}))$, for $i = 1, \cdots, N$ that are directions of linearity of T^c at $(\mathbf{X}^{c*}, \mathbf{V}^c)$;
2) for $i = 1, \cdots, N$, the vectors $\delta_i(\mathbf{p})$ are linearly independent and the same for all countries;
3) the mappings $\psi_i : \Pi \to \mathbb{R}^M$ for $i = 1, \cdots, N$ are the same for all countries.

Proof: See the Appendix. ■

Suppose that the economy has a price vector, $\mathbf{p}$, and a factor endowment of $\mathbf{V}^c \in \Gamma_N^c(\mathbf{p})$. Now imagine a change in the economy's endowment of its ith factor, $\delta_i(\mathbf{p})$. The directions of linearity, $(\psi_i(\mathbf{p}), \delta_i(\mathbf{p}))$, at each $(\mathbf{X}^{c*}, \mathbf{V}^c)$ allow the gradient vector of T^c to remain constant if the net output, $\mathbf{X}^{c*}$, changes by the amount of $\psi_i(\mathbf{p})$. Since the N independent vectors ($\delta_i(\mathbf{p})$ for $i = 1, \cdots, N$) span an N dimensional space, any change in the economy's factor endowment can be allocated to N directions of linearity. Therefore, it is possible for the economy to adjust production to any local change in its factor endowment so that the gradient vector, $\nabla T^c(\mathbf{X}^{c*}, \mathbf{V}^c)$, remains constant. Based on (24) we can see that $\partial R^c/\partial V_i = \mathbf{p}\left(\partial X^{c*}/\partial V_i\right) = W_i$. Since Theorem 5 ensures that $\partial X^{c*}/\partial V_i = \psi_i(\mathbf{p})$ is the same for all countries, the FPE holds.

Theorem 5 implies that even though the equilibrium production vectors differ from country to country (i.e. free trade is achieved), the factor prices can still be equalised. The following theorem emphasises this.

Theorem 6: Under Assumptions 2.5.1 and 2.5.2, the FPE holds for $\mathbf{p} \in \Pi$ and $(\mathbf{V}^c)_{c=1,\cdots,C} \in \underset{c=1,\cdots,C}{\times} \Gamma_N^c(\mathbf{p})$ if and only if there exist $\psi_i(\mathbf{p})$ for $i = 1, \cdots, N$ that are the same for all countries such that:

$$\nabla_{\mathbf{XV}} T^c\left(\mathbf{X}^{c*}, \mathbf{V}^c\right) = -\nabla_{\mathbf{XX}} T^c\left(\mathbf{X}^{c*}, \mathbf{V}^c\right)\mathbf{\Psi}, \tag{26}$$

$$\nabla_{\mathbf{VV}} T^c\left(\mathbf{X}^{c*}, \mathbf{V}^c\right) = \mathbf{\Psi}^T \nabla_{\mathbf{XX}} T^c\left(\mathbf{X}^{c*}, \mathbf{V}^c\right)\mathbf{\Psi}, \tag{27}$$

$$\nabla_{\mathbf{X}} T^c\left(\mathbf{X}^{c*}, \mathbf{V}^c\right)\mathbf{\Psi} + \nabla_{\mathbf{V}} T^c\left(\mathbf{X}^{c*}, \mathbf{V}^c\right)\mathbf{\Omega} = \mathbf{0}, \tag{28}$$

where $\mathbf{\Psi} \equiv \begin{bmatrix} \psi_1^1(\mathbf{p}) & \cdots & \psi_N^1(\mathbf{p}) \\ \vdots & \ddots & \vdots \\ \psi_1^M(\mathbf{p}) & \cdots & \psi_N^M(\mathbf{p}) \end{bmatrix}$, $\mathbf{\Omega}$ is an identity matrix of order N defined by $\mathbf{\Omega} \equiv (\delta_1, \cdots, \delta_N)$, the elements of which, δ_i, are the basis vectors for $i = 1, \cdots, N$, $\nabla_{\mathbf{X}} T^c(\mathbf{X}^{c*}, \mathbf{V}^c) \equiv \left[\frac{\partial T^c}{\partial X_1^c} \quad \frac{\partial T^c}{\partial X_2^c} \quad \cdots \quad \frac{\partial T^c}{\partial X_M^c}\right]$, $\nabla_{\mathbf{V}} T^c(\mathbf{X}^c, \mathbf{V}^c) \equiv \left[\frac{\partial T^c}{\partial V_1^c} \quad \frac{\partial T^c}{\partial V_2^c} \quad \cdots \quad \frac{\partial T^c}{\partial V_N^c}\right]$, and $\nabla_{\mathbf{XX}} T^c(\mathbf{X}^{c*}, \mathbf{V}^c), \nabla_{\mathbf{XV}} T^c(\mathbf{X}^{c*}, \mathbf{V}^c)$, and others denote the Hessian matrices of T^c.

Proof: See the Appendix. ■

(26) and (27) demonstrate that substitutability of the factor endowments, $\nabla_{\mathbf{VV}} T^c(\mathbf{X}^{c*}, \mathbf{V}^c)$, and the interaction between endowments and commodities $(\nabla_{\mathbf{XV}} T^c(\mathbf{X}^{c*}, \mathbf{V}^c))$ are determined by $\nabla_{\mathbf{XX}} T^c(\mathbf{X}^{c*}, \mathbf{V}^c)$ and $\psi_i(\mathbf{p})$, $i = 1, \cdots, N$. This implies that there is no restriction to the substitutability between commodities. Therefore, if the price of commodities, $\mathbf{p}$, changes, then the equilibrium production vector can vary from country to country. Since $\psi_i(\mathbf{p})$ are the same for all countries, however, the production vector's response to changes in factor endowments must be the same for every country.

Furthermore, this model can be also applied to the case wherein Π and $\Gamma^c(\mathbf{p})$ are not of full dimension, that is, when N dimensional space cannot be spanned while there are N primary factors. Let $\Gamma_K^c(\mathbf{p})$ for $\mathbf{p} \in \Pi$ denote a K dimensional subset of endowment space where $K \leqq N$. Furthermore, we assume that $\Gamma_K^c(\mathbf{p})$ is convex.

Definition 2.5.4: The vectors $\delta_i(\mathbf{p})$ for $i = 1, \cdots, K$ are said to locally span $\Gamma_K^c(\mathbf{p})$ at $\mathbf{V}^c \in \Gamma_K^c(\mathbf{p})$ if they span the K dimensional affine subset that contains $\Gamma_K^c(\mathbf{p})$.

As is shown by Theorems 5 and 6, and given the above concept, factor prices equalise in the K dimensional subspace (Blackorby et al., 1993).

Blackorby et al. (1993) is more general than Kuga (1972) and Mas-Collel (1979a, b) in that the assumption of production technique allows for decreasing returns to scale, joint production, and the existence of inputs other than primary factors (i.e. intermediate goods). In order to derive the condition for factor price equalisation, it is crucial that the transformation functions, T^c, are directions of linearity. Kuga (1972) is similar to Blackorby et al.'s (1993) in that they both allow the number of final commodities to differ from the number of primary factors and both of their conditions for factor price equalisation hold that factor prices are solely dependent on commodity prices and independent of factor endowments. It should be noted, however, that Kuga's definition of factor price equalisation differs slightly from that used by Blackorby et al.; specifically, the former is stronger than the latter. Additionally, both models rely on different mechanisms to equalise factor prices. Kuga (1972) uses the method presented in Theorem 3 because the direct effect would be offset by the adjustment effect, not because the gradient vector of the transformation function would remain constant in the face of changing factor endowments. The necessary and sufficient condition derived from Blackorby et al. (1993) is weaker than Kuga (1972) in that the class of production economies supposed in the former is broader than that in the latter and the definition of factor price equalisation is weaker.

It is unclear how broad the class of production economies that satisfy Blackorby et al. (1993) necessary and sufficient conditions are. However, we can check whether or not the factor prices equalise in a production economy by using Theorem 5.

3 The HOS Model with Reproducible Capital

In the previous section, capital is a primary factor of production. As the classical economists and Marx emphasised, however, the capitalist economic system has been based on the establishment of industrialised societies since the 19th century. An essential feature of this system is that capital follows a circuit between a monetary form of value and heterogeneously reproducible commodities.

The Cambridge capital controversies revealed that it would be problematic to introduce the idea of capital as a bundle of reproducible commodities into neo-classical economic theory. These controversies arose in the 1960's and 1970's between the neo-classical economists who resided mainly in Cambridge, Mass. (e.g. Samuelson, Solow, Modigliani, Burmeister, Meade, and Hahn) and those who resided in Cambridge, UK (J. Robinson, Pasinetti, Garegnani, Kaldor, and Sraffa). The primary sources of the controversies were the concept of capital, the logical validity of the neo-classical production function and principle of marginal productivity.

The controversies brought to light several problematic issues with the HOS model, namely the questions of whether 'reswitching of techniques' or 'capital reversing' could occur when capital is taken as a bundle of reproducible commodities. The former is a phenomenon in which one technique could correspond to some rate of profit and the latter is that the decreasing monotonicity between the rate of profit and the capital intensity would not necessarily hold. These phenomena imply that the properties of the neo-classical cost function may not hold.

As Sraffa (1960) shows, when capital consists of a bundle of reproducible commodities, the price of capital must be determined simultaneously with the price structure and the rate of profit, and as such, the capital endowment cannot be formulated independently of the income distribution, like $\mathbf{V}$ in the previous section. When countries are allowed to choose their techniques, one technique might correspond to some rate of profit. If the rate of profit is regarded as the factor price, then it would imply that there is a possibility that the commodity price and the factor price are not global univalent. Therefore, it is unclear whether or not factor price equalisation can still be characterised by the theorems described in the previous section when capital consists of a bundle of reproducible commodities.

3.1 The Cambridge Capital Controversies

First, let us briefly review the most relevant issues highlighted by the Cambridge controversies.[17]

This paper will focus on 'capital reversing'. According to the neo-classical production function, the wage-profit curve (or factor price frontier) is convex toward the origin, there is a one-to-one correspondence between the rate of profit and a technique, and capital intensity is monotonically decreasing with respect to the rate of profit (i.e. the marginal productivity of capital principal is at work). In the neo-classical production function, capital is, of course, a primary factor.

Samuelson (1962) attempts to apply the aforementioned properties of the neo-classical production function to the case where capital consists of a bundle of reproducible commodities. He constructs a simple model in which one kind of consumption commodity is produced by using labour and capital, which is, itself, reproducible. This technique is characterised by fixed coefficients. Moreover, this model assumes capital to be heterogeneous; it is indexed by capital $\alpha, \beta, \gamma \cdots$. Therefore, we cannot produce capital, α, by using labour and other capital, like $\beta, \gamma \cdots$. Furthermore, it is assumed that the capital-labour ratio used to produce capital, α, is technologically given and is the same as the capital-labour ratio for the consumption commodity when it is produced by using capital α. The capital-labour ratio to reproduce capital β is different from the ratio to reproduce capital α, but it is the same as the capital-labour ratio for the consumption commodity when it is produced by using capital β. The same assumption is imposed on the reproduction of all capital.

In this case, the wage-profit curves that correspond to the different capitals become straight. This is because, thanks to the assumptions that the techniques are represented by fixed coefficients and both consumption commodity and capital require the same capital-labour ratio, the price structure remains unaffected by changes in the income distribution. The technological relationship concerning income distribution is described by the envelope of all the straight wage-profit lines. The envelope is convex to the origin.

Samuelson concludes that the envelope, which he obtains from heterogeneous capital, could sufficiently approximate the wage-profit curve that is obtained if one assumes that capital is a primary factor. The approximation of the production function is termed the 'surrogate production function', and the approximated capital is termed 'surrogate capital'.

If Samuelson were correct, then the principle of marginal productivity of capital could be utilised even if capital were heterogeneous. This is because there is a one-to-one correspondence between the rate of profit and the technique. The 'non-switching' theorem proven by Levhari (1965) holds that one technique would

[17] The controversies covered a wide range issues; see Blaug (1975), Cohen and Harcourt (2003), Harcourt (1972), and Pasinetti (2000) for further details on their scope.

not correspond to the same rates of profit throughout the entire economic system, which seems to further support Samuelson's conclusion.

However, Samuelson's conclusion relies heavily on the assumption that the capital-labour ratio of capital production is the same as that of consumption production, which is a singularly peculiar assumption. Pasinetti (1966) is the first to produce a counter-example to Levhari (1965)'s non-switching theorem; as a result, it is made clear that the surrogate production function had no general foundation for economic analysis.[18] Without the assumption, capital reversing can occur; in other words, capital intensity may not be a monotonically decreasing function of the rate of profit, which contradicts the principle of marginal productivity.

Moreover, some rates of profit may correspond to one technique, which is termed the 'reswitching of techniques'. Suppose that α and β are alternative techniques available in an economic system and r denotes the rate of profit; the reswitching of techniques occurs if α is the cost minimising technique at $r \in [0, r_1]$, β is the cost minimising technique at $r \in [r_1, r_2]$ where $r_1 < r_2$, and α is the cost minimising technique at $r \in [r_2, R_\alpha]$ where R_α is the maximum rate of profit under technique α.[19] This is also a phenomenon that is inconsistent with the principle of marginal productivity. Moreover, it is made clear that the reswitching of techniques can take place regardless of whether the techniques are decomposable or not.[20]

The Cambridge capital controversies provide sufficient reason to doubt the validity of theorems derived from the HOS model, which relies on a cost function derived from the neo-classical production function.

3.2 The Model with Reproducible Capital: the case of $n = 2$

In the 1970s and 1980s, the HOS model was criticised by Mainwaring, Metcalfe, and Steedman exclusively on the basis of the critiques raised by the controversies (Metcalfe and Steedman, 1972, 1973; Mainwaring, 1984; Steedman, 1979). Although they present a number of numerical examples and criticised the FPET, they do not rigorously argue the (in)validity of the FPET under the Leontief production model. Consequently, we shall derive a theorem that states that the FPET is valid under the Leontief production model with $n = 2$. In this case, however, the necessary and sufficient conditions for factor price equalisation will be extremely restrictive. Therefore, we shall show that the theorem's implications may be interpreted as the impossibility theorem.

The basic premise of the model is that labour is only a primary factor and that all physical input is composed of reproducible commodities. One technique is represented by the Leontief production model; consequently, the equilibrium price of commodity j $(j = 1, \cdots, n)$ is given as follows:

$$p_j = l_j w + (1 + r) \sum_{i=1}^{n} a_{ij} p_i,$$

where $l_j > 0, a_{ij} \geqq 0, w \geqq 0, r \geqq 0$ denote the labour coefficient, physical coefficient, the wage rate, and the rate of profit, respectively. For simplicity, we assume that capital is circulating. In general, there are some Leontief techniques available for the production of commodity j. The criterion for the choice of techniques is that it minimise the production cost given a certain price system. Suppose that the following is the cost minimising technique under price system $(\mathbf{p}, w, r)$:

[18] See the set of papers published in the 'Paradoxes in Capital Theory' symposium in the *Quarterly Journal of Economics*: Bruno et al. (1966), Garegnani (1966), Levhari and Samuelson (1966), Morishima (1966), Pasinetti (1966), Samuelson (1966b).

[19] See Pasinetti (1977, chap. 6). There were three types of reactions from neo-classical economics against the critiques levelled by the neo-Ricardians; the first was to describe the phenomenon as a 'paradox', 'perverse', 'exceptional', 'inconvenient', or 'anomalous', that is, contending that the phenomenon was rarely observed in reality and therefore irrelevant (Blaug, 1975; Samuelson, 1966b); the second was to attempt to investigate the conditions under which capital reversing or reswitching of techniques would not take place (Burmeister and Dobell, 1970; Burmeister, 1980); the third was to assert that the neo-Ricardian model was merely a special case of the intertemporal general equilibrium model and that it could consequently be freed from the neo-Ricardian critiques (Hahn, 1982). See Pasinetti (2000) concerning this topic in detail.

[20] Burmeister (1980, pp. 114–115) asserts that the reswitching of techniques is an irrelevant phenomenon within the field of neo-classical economics. This is because it does not necessarily accompany the paradoxical behaviour of consumption. Neo-classical economic thought maintains that the steady state consumption level per capita is a monotonically decreasing function of the rate of profit. This simple relation is a parable derived from a one-commodity model. According to Burmeister, any phenomenon which is not inconsistent with the neo-classical parable does not vitiate the essence of neo-classical economics.

$$\left(\left(a_{ij} \left(\mathbf{p}, w, r \right) \right)_{i=1,\ldots,n}, l_j \left(\mathbf{p}, w, r \right) \right) \text{ for } \forall j = 1, \cdots, n.$$

Then, the following theorem is valid.

Theorem 7: In the case of $n = 2$ under the Leontief production model, the commodity price and the rate of profit are global univalent if and only if there is no capital intensity reversal.

Proof: See the Appendix. ■

The relationship between the wage-profit curve and relative price is such that:

$$\frac{\mathrm{d}p}{\mathrm{d}r} = -\frac{l_1 \left(\mathbf{p}, w, r \right) \left\{ 1 - (1 + r) a_{22} \left(\mathbf{p}, w, r \right) + (1 + r) l_2 \left(\mathbf{p}, w, r \right) a_{21} \left(\mathbf{p}, w, r \right) \right\}}{2 a_{21} \left(\mathbf{p}, w, r \right)} \frac{\mathrm{d}^2 w^1}{\mathrm{d}r^2},$$

which implies:

$$sign \left(\frac{\mathrm{d}^2 w^1}{\mathrm{d}r^2} \right) = -sign \left(\frac{\mathrm{d}p}{\mathrm{d}r} \right).$$

As Mainwaring (1984) points out, the relative capital intensity determines the sign of $\frac{\mathrm{d}p}{\mathrm{d}r}$, which in turn, determines the form of the wage-profit curve as far as the two-commodity Leontief production model is concerned. In the two-commodity Leontief production model, if the numéraire industry is more capital intensive (labour intensive) than the other industry, then the relative price will be a decreasing (increasing) function of the rate of profit and the wage-profit curve will be concave (convex) to the origin.

Under a convex production set, the technical change that reverses the size of $\frac{l_1(\mathbf{p},w,r)a_{12}(\mathbf{p},w,r)+l_2(\mathbf{p},w,r)a_{22}(\mathbf{p},w,r)}{l_2(\mathbf{p},w,r)}$ and $\frac{l_1(\mathbf{p},w,r)a_{11}(\mathbf{p},w,r)+l_2(\mathbf{p},w,r)a_{21}(\mathbf{p},w,r)}{l_1(\mathbf{p},w,r)}$ is not peculiar at all. Therefore, in some limited cases, factor intensity reversal may not occur. Therefore, we may interpret Theorem 7 as a *de facto* impossibility theorem of factor price equalisation.

3.3 The Neo-Ricardian Critique of the HOS Model

The previous subsection demonstrates that factor intensity could be easily reversed when capital consisted of reproducible commodities. This leads us to wonder whether we can show that factor prices will not necessarily equalise in the absence of factor intensity reversals if capital consists of a bundle of reproducible commodities.

In order to answer this question, we utilise a two-integrated-sector model.[21] Both Sectors 1 and 2 are composed of consumption good and capital good producing industries. Sector 1's consumption good producing industry is termed Industry 1 and its capital good producing industry is Industry 2; similarly, the consumption good producing industry of Sector 2 is Industry 3, and the capital good producing industry of Sector 2 is Industry 4.

Let us assume that Industry 1 has three available techniques:

$$\left(a_{11}^{\alpha}, a_{21}^{\alpha}, l_1^{\alpha} \right) = (0.38, 0.63, 0.06),$$
$$\left(a_{11}^{\beta}, a_{21}^{\beta}, l_1^{\beta} \right) = (0.4188, 0.424, 0.265),$$
$$\left(a_{11}^{\gamma}, a_{21}^{\gamma}, l_1^{\gamma} \right) = (0.52, 0.01, 0.65).$$

$a_{ij}^{\iota}, l_j^{\iota}$ denote the amount of commodity i and labour that are required to produce a unit of commodity j under technique ι ($\iota = \alpha, \beta, \gamma$). On the other hand, Industry 2 has only one available technique:

$$\left(a_{12}, a_{22}, l_2 \right) = (0.08, 0, 1).$$

[21] Takamasu (1991) attempts to criticise the FPET by using a numerical example. Unfortunately, however, the capital intensity reversal occurs in his example. Therefore, he fails to criticise the FPET. Our forthcoming numerical example is a modification of his example.

Figure 1 depicts the envelope of the wage-profit curves that were obtained under each technique. The vertical axis of the figure represents the wage rate in terms of the commodity produced by Industry 1, w_1.

Insert Figure 1 here.

There are four switching points; technique α is chosen if $0 \leqq r \leqq r_1 \cong 0.18$; technique β is chosen if $r_1 \leqq r \leqq r_2 \cong 0.317$; technique γ is chosen if $r_2 \leqq r \leqq r_3 \cong 0.503$; technique β is chosen again if $r_3 \leqq r \leqq r_4 \cong 0.9003$;and technique α is chosen again if $r_4 \leqq r \leqq R_\alpha \cong 1.066$, where R_α is the maximum rate of profit in Sector 1. The reswitching of techniques occurs.

Let $w_1^\iota(r)$ and $k_1(r)$ denote the wage rate measured by the consumption commodity produced in Sector 1 under technique ι and the capital intensity in terms of the consumption good, respectively. $k_1(r)$ is defined as follows:[22]

$$k_1(r) = \begin{cases} \left| \frac{\mathrm{d}w_1^\alpha(r)}{\mathrm{d}r} \right|_{r=0} \Big|, & \text{if } r = 0, \\ \frac{w_1^\alpha(0) - w_1^\alpha(r)}{r}, & \text{if } 0 < r \leqq r_1 \text{ and } r_4 \leqq r \leqq R_\alpha, \\ \frac{w_1^\beta(0) - w_1^\beta(r)}{r}, & \text{if } r_1 \leqq r \leqq r_2 \text{ and } r_3 \leqq r \leqq r_4, \\ \frac{w_1^\gamma(0) - w_1^\gamma(r)}{r}, & \text{if } r_2 \leqq r \leqq r_3, \end{cases} \tag{29}$$

The switch from γ to β at $r = r_3$ and that from β to α at $r = r_4$ does not adhere to the monotonically decreasing relationship between the rate of profit and capital intensity; in other words, there is capital reversing.

With respect to Sector 2, let us assume that Industry 3 has two technique alternatives:

$$\left(a_{33}^\delta, a_{43}^\delta, L_3^\delta\right) = (0.2, 0.485, 0.03),$$
$$\left(a_{33}^\epsilon, a_{43}^\epsilon, L_3^\epsilon\right) = (0.3, 0.41, 0.02).$$

On the other hand, Industry 4 has only one available technique:

$$(a_{34}, a_{44}, L_4) = (0.29, 0, 1.61).$$

Letting w_2 be the wage rate in terms of Sector 2's consumption commodity (the product of Industry 3), Figure 2 depicts the wage-profit curves.

Insert Figure 2 here.

In Sector 2, ε switches to δ at $r = r_5 \cong 0.205$ but the techniques do not reswitch and there is no capital reversing. Using the same procedure as described in (29), we can obtain capital intensity in terms of Sector 2's consumption commodity, which is denoted as $\overline{k}_2(r)$.

Let p_1 and p_2 be the price of Sector 1's and Sector 2's consumption commodities, respectively. In order to compare the capital intensity of both sectors, it must first be measured in terms of the same commodity price. Let $k_2(r)$ be the capital intensity of Sector 2 in terms of Sector 1's consumption commodity. Then, $k_2(r) = \overline{k}_2(r) \times \frac{p_2}{p_1}$ is, by definition, given. Since the wage rate is uniform in both sectors, $\frac{p_2}{p_1} = \frac{w_1}{w_2}$, which implies that $k_2(r) = \overline{k}_2(r) \times \frac{w_1}{w_2}$.

Table 1 presents a summary of the above model.

Insert Table 1 here.

[22] See Garegnani (1970).

Table 1 shows that Sector 2 is always more capital intensive than Sector 1, which means that no capital intensity reversal occurs in this model (see Figure 3 as well). Moreover, it shows that the relative price, p_2/p_1, is not a monotonic function of the rate of profit (see Figure 4). This means that the relative price and the rate of profit are not global univalent.

Insert Figures 3 and 4 here.

In other words, the FPET does not necessarily hold if capital consists of a bundle of reproducible commodities. This is a problem that neo-classical economists, who treat capital merely as a primary factor, cannot neglect.

Our numerical example featured four commodities, two of which are, so to speak, the intermediate goods, while the other two are consumption goods that are usually traded in the international market. Here, we argue that, with respect to the two consumption goods, there is global univalence between the relative price and the rate of profit. In this sense, the numerical example is essentially equivalent to a two-good, two-factor model (the factor is not primary but reproducible here). The setting of the numerical example is parallel to the model used in the proof of Theorem 7. Notwithstanding, if capital consists of a bundle of reproducible commodities, then the global univalence between the relative price and the rate of profit may not necessarily be ensured.

Note that Dixit (1981, pp. 291-292) argues that the existence of non-tradable goods as circulating capital inputs other than the two tradable, final consumption goods causes the impossibility to derive a 'simple condition'[23] ensuring the univalent relation between prices of commodities and factor prices, since the expression for the elasticity of a wage and profit rate frontier involves indirect effects working through induced changes in prices of non-tradable goods. In comparison to Dixit's (1981) claim, what we have discussed above provides a strengthening of the impossibility of the univalence relation, since the standard condition of no capital intensity reversal is satisfied by the economy constructed in this subsection. It may suggest a more fundamental source of the impossibility of the univalence relation, which remains to be a future agenda.

4 The HOS Model after the Controversies

In regard to the introduction of capital as a bundle of reproducible commodities into the HOS model, Samuelson first argues:

> Now suppose there are uniform differences in factor intensity, so that for some two goods that are simultaneously produced in both countries, say goods 1 and 2, $p_1(r)/p_2(r) = p_{12}(r)$ is a monotone, strictly increasing (or decreasing) function of r [the interest rate]. *Then, the interest rate will be equalized by positive trade in those goods alone*' (Samuelson, 1965, p. 49).

Bliss (1967) criticises Samuelson (1965), arguing that the problem is the condition for the monotonic relationship between the relative price and the rate of profit. However, Samuelson said nothing of this.[24]

In light of Bliss' (1967) and Metcalfe and Steedman's (1972, 1973) critiques, Samuelson (1975) acknowledges the possibility that the FPET might not hold globally but rather just locally when capital consists of a bundle of reproducible commodities (i.e. the local factor equalisation theorem). However, Samuelson (1975, p. 351) believes that Metcalfe and Steedman's (1972, 1973) warning is non-academic and thus gives little credence to it. Just as the neo-classicals argue about the Cambridge capital controversies, he contends that the phenomena described by Metcalfe and Steedman are unlikely to occur in reality.

Following the neo-Ricardian critiques, Burmeister (1978) constructs the most rigorous model with reproducible capital.

[23] Here, the simple condition seems to be the standard condition of no capital intensity reversal.
[24] See Samuelson (1978) for more on this point.

4.1 Burmeister (1978)

As Samuelson frankly admits, the FPET does not necessarily hold if capital consists of a bundle of reproducible commodities. Therefore, stronger conditions must be imposed on the model in order for the FPET to hold.

Burmeister (1978) specifies the conditions by using a P-matrix, which generalised the Stolper–Samuelson Theorem put forward by Chipman (1969), Inada (1971), and others. Inada (1971) modifies the SSS condition defined in Section 2.2 as follows:

SSS-I Condition: All the diagonal elements of $\mathbf{A}^{-1}$ are positive and all the non-diagonal elements are negative.

SSS-II Condition: All the diagonal elements of $\mathbf{A}^{-1}$ are negative and all the non-diagonal elements are positive.

Here, $\mathbf{A}$ has no restriction except a square and non-negative matrix, while the SSS condition in Section 2.2 is characterised by $\widetilde{\mathbf{A}}^{-1} = \left[\alpha^{ij}\right]$.

Burmeister (1978) assumes that there are m consumption goods, n reproducible capital goods, and h primary factors, where $h \leqq m$. First, let us consider an economy in which there is no opportunity to choose techniques. The rate of profit is denoted by r, the capital good price vector by $\mathbf{p} \equiv [p_i]$ $(i = 1, 2, \cdots n)$, the consumption good price vector by $\mathbf{s} \equiv [s_i]$ $(i = n+1, n+2, \cdots, n+m)$, and the primary factor price vector by $\mathbf{w} \equiv [w_i]$ $(i = 1, 2, \cdots, h)$. Furthermore, the capital coefficient matrix is represented by $\mathbf{A} \equiv \begin{bmatrix} a_{11} & \cdots & a_{1n} & a_{1,n+1} & \cdots & a_{1,n+m} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} & a_{n,n+1} & \cdots & a_{n,n+m} \end{bmatrix}$, and the primary factor coefficient matrix is represented by $\mathbf{e} \equiv \begin{bmatrix} e_{11} & \cdots & e_{1n} & e_{1,n+1} & \cdots & e_{1,n+m} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ e_{h1} & \cdots & e_{hn} & e_{h,n+1} & \cdots & e_{h,n+m} \end{bmatrix}$.

Consequently, the price equation is given as follows:

$$[\mathbf{p}, \mathbf{s}] = \mathbf{we} + (1 + r)\,\mathbf{pA}, \tag{30}$$

where $[\mathbf{p}, \mathbf{s}] = [p_1, \cdots, p_n, s_{n+1}, \cdots, s_{n+m}]$. It is assumed that m consumption goods and at least one capital good are freely traded internationally and that consumption good 1 is adopted as the numéraire $(s_{n+1} = 1)$.

Following Sraffa's (1960) terminology, consumption goods are the 'non-basic' goods under the above assumptions, and as such, the production condition of those 'non-basic' goods has no effect on the rate of profit or the prices of the 'basic' goods.[25] This implies that, without losing generality, we can assume that $m = h$. As such, matrices $\mathbf{A}$ and $\mathbf{e}$ can be compiled as follows:

$$\overline{\mathbf{A}} \equiv \begin{bmatrix} a_{11} & \cdots & a_{1n} & a_{1,n+1} & \cdots & a_{1,n+h} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} & a_{n,n+1} & \cdots & a_{n,n+h} \\ e_{11} & \cdots & e_{1n} & e_{1,n+1} & \cdots & e_{1,n+h} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ e_{h1} & \cdots & e_{hn} & e_{h,n+1} & \cdots & e_{h,n+h} \end{bmatrix} = \begin{bmatrix} \overline{\mathbf{A}}_1 & \overline{\mathbf{A}}_2 \\ \overline{\mathbf{A}}_3 & \overline{\mathbf{A}}_4 \end{bmatrix}.$$

Let us assume that $\overline{\mathbf{A}}$ is non-singular and defined as follows:

$$\mathbf{B} \equiv \overline{\mathbf{A}}^{-1} = \begin{bmatrix} \mathbf{B}_1 & \mathbf{B}_2 \\ \mathbf{B}_3 & \mathbf{B}_4 \end{bmatrix}.$$

[25] In order to properly analyse techniques and income distributions, it is important to clearly distinguish between basic and non-basic goods. According to Sraffa (1960), a 'basic' good is a commodity that is directly or indirectly required for the production all commodities, while 'non-basic' goods encompass all other commodities.

$\mathbf{B}_1, \mathbf{B}_2, \mathbf{B}_3, \mathbf{B}_4$ denote a square matrix of order n, a $(n \times h)$ matrix, a $(h \times n)$ matrix, and a square matrix of order h, respectively.

Letting $\overline{\mathbf{s}} \equiv \left[1, \frac{s_{n+2}}{s_{n+1}}, \cdots, \frac{s_{n+h}}{s_{n+1}}\right]$ be the consumption good price vector when $s_{n+1} = 1$ allows us to rewrite (30) as follows:

$$[\mathbf{p}, \overline{\mathbf{s}}] = [(1+r)\,\mathbf{p}, \mathbf{w}]\,\overline{\mathbf{A}}, \text{ or} \tag{31}$$

$$[\mathbf{p}, \overline{\mathbf{s}}]\,\overline{\mathbf{A}}^{-1} = [\mathbf{p}, \overline{\mathbf{s}}]\,\mathbf{B} = [(1+r)\,\mathbf{p}, \mathbf{w}]\,. \tag{32}$$

Here, $\overline{\mathbf{s}}$ is regarded as a given vector, as it is assumed to be determined by free trade. Because of the assumption that $\overline{\mathbf{A}}$ is non-singular, $\overline{\mathbf{s}}\mathbf{B}_3 = \mathbf{w}$ holds, which implies that primary factor prices will equalise.

By transforming the first n equations of (32), we obtain:

$$\mathbf{p}\,[\mathbf{B}_1 - (1+r)\,\mathbf{I}] = -\overline{\mathbf{s}}\mathbf{B}_3, \tag{33}$$

where $\mathbf{I}$ is an identity matrix of order n. Because of (33), we can confirm that, given the consumption good price vector, the capital good price vector uniquely determines the rate of profit if the price of capital goods is a monotonic function of the rate of profit, which validates the FPET. As such, we obtain the following theorem:

Theorem 8: If the production of m capital goods and h ($\leqq m$) consumption goods is satisfied by the SSS-II (the SSS-I) condition, then $\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}r} < (>)\,0$.

Proof: See the Appendix. ■

Theorem 8 implies that, given $\overline{\mathbf{A}}$ and the consumption good price vector, the capital good price vector is a monotonic function of the rate of profit if the SSS-I or SSS-II condition is satisfied. Under this condition, the prices of not only primary factors but also of capital goods are equalised.

Next, let us consider an economy in which there is a choice of techniques, as in the neo-classical production function. The relationship between factor rent, q_i, and capital good prices, p_i, can be obtained in equilibrium: $p_i = \frac{q_i}{1+r}$. Differentiating (31) with respect to r yields:

$$\left[\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}r}, \frac{\mathrm{d}\overline{\mathbf{s}}}{\mathrm{d}r}\right] = \left[\frac{\mathrm{d}\mathbf{q}}{\mathrm{d}r}, \frac{\mathrm{d}\mathbf{w}}{\mathrm{d}r}\right]\overline{\mathbf{A}} + [\mathbf{q}, \mathbf{w}]\,\frac{\mathrm{d}\overline{\mathbf{A}}}{\mathrm{d}r},$$

where $\mathbf{q} \equiv [q_i]$ $(i = 1, 2, \cdots, n)$. Since we assume the neo-classical production function, $[\mathbf{q}, \mathbf{w}]\,\frac{\mathrm{d}\overline{\mathbf{A}}}{\mathrm{d}r} = \mathbf{0}$ holds; therefore, we obtain:

$$\left[\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}r}, \frac{\mathrm{d}\overline{\mathbf{s}}}{\mathrm{d}r}\right]\mathbf{B} = \left[\frac{\mathrm{d}\mathbf{q}}{\mathrm{d}r}, \frac{\mathrm{d}\mathbf{w}}{\mathrm{d}r}\right]. \tag{34}$$

We assume that $\mathbf{B} = \overline{\mathbf{A}}^{-1}$ is ensured for all the possible technique choices. In other words, we assume that while the price structure changes as the optimally chosen techniques change (and thus the elements of $\overline{\mathbf{A}}$ change as well), the change in techniques is limited only in cases where $\overline{\mathbf{A}}$ is non-singular.

As before, we can confirm that, given the consumption good price vector, the capital good price vector uniquely determines the rate of profit if the price of capital goods is a monotonic function of the rate of profit. If we can confirm this, then we can say that the prices of capital goods equalise even when there is a choice of techniques. Consequently, we obtain the following theorem:

Theorem 9: If all countries produce n capital goods and h ($\leqq m$) consumption goods subject to the neo-classical production function (4), and if the SSS-II (or SSS-I) condition is satisfied at every feasible factor price $(q_1, \cdots, q_n, w_1, \cdots, w_h)$ where $q_i = (1+r)\,p_i$, then $\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}r} < (>)\,\mathbf{0}$.

Proof: See the Appendix. ■

Theorem 9 implies that, given the consumption good price vector, the capital good price vector is a monotonic function of the rate of profit if capital goods are produced on the basis of the neo-classical production function and the SSS-I or SSS-II condition is satisfied. This means that there is a one-one correspondence between the capital good prices and capital rental rates, and thus, capital good rental rates are internationally equalised under the equilibrium price system. On the other hand, due to the implicit assumption that $\mathbf{B}_3$ always exists, there is a one-to-one correspondence between the primary factor price vector and the consumption good price vector, and as such, factor prices equalise.

On first inspection, the economic meaning of the SSS-I and SSS-II conditions seems unclear. The SSS-I condition implies that $\mathbf{A}^{-1}$ is a Minkowski matrix, and the SSS-II condition implies that $\mathbf{A}^{-1}$ is a Metzler matrix. As Uekawa et al. (1972) point out, the SSS-I and the SSS-II conditions are equivalent to the SSS-I$'$ and the SSS-II$''$ conditions, respectively.

SSS-I$'$ Condition: The inverse of the non-negative matrix $\mathbf{A} \equiv [a_{ij}]$ is a Minkowski matrix if and only if, for any non-empty proper subset J of $N = \{1, 2, \cdots, n\}$ and any given $\overline{x}_i > 0, i \in J^C$, there exists $x_i > 0, i \in J$, such that

$$\sum_{i \in J} a_{ij} x_i > \sum_{i \in J^C} a_{ij} \overline{x}_i \text{ for } j \in J,$$
$$\sum_{i \in J} a_{ij} x_i < \sum_{i \in J^C} a_{ij} \overline{x}_i \text{ for } j \in J^C.$$

SSS-II$'$ Condition: The inverse of the non-negative matrix $\mathbf{A} \equiv [a_{ij}]$ is a Metzler matrix if and only if, for $J \subset N$ and any given $\overline{w}_j > 0, j \in J^C$, there exists $w_j > 0, j \in J$, such that

$$\sum_{j \in J} w_j a_{ij} < \sum_{j \in J^C} \overline{w}_j a_{ij} \text{ for } i \in J,$$
$$\sum_{j \in J} w_j a_{ij} > \sum_{j \in J^C} \overline{w}_j a_{ij} \text{ for } i \in J^C.$$

According to Uekawa et al. (1972), the SSS-I$'$ and SSS-II$'$ conditions bear the following economic implications:

SSS-I$'$ Condition: Suppose that commodities are randomly grouped into two composite commodities, J and J^C, and let x_i be the output of the ith commodity. Then, for any non-trivial J and any set of outputs $\overline{x}_i > 0, i \in J^C$, there exists $x_i > 0, i \in J$, such that more of the jth factor ($j \in J$) and less of the jth factor ($j \in J^C$) is used to produce the composite commodity J than to produce J^C.

SSS-II$'$ Condition: Suppose that the primary factors are randomly grouped into two composite factors, J and J^C, and let w_j be the price of factor j. Then, for any non-trivial J and any set of factor prices, $\overline{w}_j > 0, j \in J^C$, there exists $w_j > 0, j \in J$, such that the composite factor J contributes less to the cost of production of the jth commodity ($j \in J$) and more to the cost of production of the jth commodity, $j \in J^C$.

Therefore, the SSS-I and SSS-II conditions characterise the factor intensity. Clearly, these conditions are extremely strong.

Unlike the traditional HOS model, Theorem 9 treats capital as a bundle of reproducible commodities. This begs the question as to how Burmeister's model is related to the outcome of the Cambridge capital controversies, which we can evaluate by simplifying the model.

The simplest case of the model features one consumption good, one capital good, and one primary factor (i.e. labour).[26] In this case, (31) is rewritten as follows:

[26] In fact, the modern dynamic HOS model features the same structure. See, for example, Chen (1992), Nishimura and Shimomura (2002, 2006), and Bond et al. (2011, 2012).

$$[p, s] = [(1+r)\,p, w]\,\overline{\mathbf{A}}, \tag{35}$$
$$\overline{\mathbf{A}} \equiv \begin{bmatrix} a_{11} & a_{12} \\ l_1 & l_2 \end{bmatrix}.$$

Despite the fact that capital goods are reproducible, (35) is a *de facto* one-good model with respect to the determination of factor prices because the consumption goods are non-basic goods. Since $\frac{\mathrm{d}p}{\mathrm{d}r} > 0$ holds because of (35), the model maintains a one-to-one correspondence between the rate of profit (or factor rental) and the price of capital goods, given the price of consumption goods.[27] This implies that the factor prices equalise. However, it is obvious that the simplified model simply avoids the difficulties pointed out by the Cambridge capital controversies because it is a *de facto* one-good model.

In other words, Burmeister's (1978) model is structured so as to circumvent the issues pointed out in the Cambridge capital controversies as it assumed away several economic environments. For instance, unlike in the Leontief production model, there are never reproducible goods, like corn, which can be used as both capital and consumption goods.

The simplest example in which there exist commodities that can be used as both capital and consumption goods is a two-good economy in which both commodities are basic goods. In this case, we obtain:

$$\mathbf{A} \equiv \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} > \mathbf{0};\ \mathbf{L} \equiv (l_1, l_2) > \mathbf{0}.$$

By applying Burmeister (1978) to this case, we find that $\overline{\mathbf{A}}$ is given as follows:

$$\overline{\mathbf{A}} \equiv \begin{bmatrix} a_{11} & a_{12} & 0 \\ a_{21} & a_{22} & 0 \\ l_1 & l_2 & 0 \end{bmatrix}.$$

Here, $\det \overline{\mathbf{A}} = 0$, which means that $\overline{\mathbf{A}}$ is singular. Therefore, we cannot obtain the property that is ensured by Theorems 8 and 9.

5 Concluding Remarks

In this paper, we surveyed the HOS model in order to determine how it was modified in reaction to the neo-classical economists' critiques in the Cambridge capital controversies.

Burmeister (1978) makes one of the most significant contributions to the HOS model in light of the controversies. He derives the conditions for factor price equalisation under the assumption that there exist reproducible capital goods. The modern dynamic HOS models that feature reproducible capital goods, such as Chen (1992), Nishimura and Shimomura (2002, 2006), and Bond et al. (2011, 2012), have essentially the same structure as Burmeister (1978). As previously mentioned, these models use assumptions to avoid the phenomena which the controversies highlighted, and thus, they exclude many economic environments that could arise in reality.

If there is no room to consider capital as a bundle of reproducible commodities rather than a primary factor, then the validity of the HOS model should be carefully reconsidered. First, if capital is treated as a bundle of reproducible commodities, then capital intensity reversal may easily arise. Second, even if there is no reversal of capital intensity, the global univalence between the rate of profit and the relative price may not hold. In other words, even in a two-good model, the FPET may not necessarily hold if capital is treated as a bundle of reproducible commodities. All of these arguments suggest that it is necessary to construct a basic theory of international trade that does not rely on factor price equalisation and treats capital as a bundle of reproducible inputs.

[27] In this case, not only the price of capital goods but also that of consumption goods has a one-to-one correspondence with the rate of profit. Due to $p = \frac{wl_1}{1-(1+r)a_{11}}$, we obtain:

$$\frac{\mathrm{d}s}{\mathrm{d}r} = \frac{wl_1 a_{12}\,[1-(1+r)\,a_{11}] + (1+r)\,wl_1 a_{12} a_{11}}{[1-(1+r)\,a_{11}]^2} > 0.$$

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7 The Appendix

In the Appendix, we show the rigorous proofs of the theorems and a numerical example.

7.1 The Proof of the FPET in the two-good and two-country model

Proof: In order to prove the global univalence of the cost function, c, it is sufficient to show that, for price equation $\mathbf{p} = \mathbf{wA}(\mathbf{w})$, the relative price, $p_1'(\mathbf{w}) \equiv \frac{p_1}{p_2}$, and $\mathbf{w}$ are global univalent. In other words, when the factor prices change in such a way that $\mathbf{w} \to \mathbf{w} + \triangle\, \mathbf{w} \equiv (w_1 + \triangle\, w, w_2 - \triangle\, w)$ for $\triangle\, w > 0$, it must be confirmed that $p_1'(\mathbf{w})$ and $\mathbf{w}$ are global univalent if $p_1'(\mathbf{w}) \to p_1'(\mathbf{w} + \triangle\, \mathbf{w})$ is monotonic. It is necessary to show that $\frac{\partial p_1'(\mathbf{w})}{\partial w_1} \triangle\, w - \frac{\partial p_1'(\mathbf{w})}{\partial w_2} \triangle\, w > 0$ or $\frac{\partial p_1'(\mathbf{w})}{\partial w_1} \triangle\, w - \frac{\partial p_1'(\mathbf{w})}{\partial w_2} \triangle\, w < 0$ always holds for $\forall p_1'(\mathbf{w})$. Thanks to

$$\frac{\partial p_1'(\mathbf{w})}{\partial w_1} = \frac{a_{11}(\mathbf{w}) p_2 - p_1 a_{12}(\mathbf{w})}{(p_2)^2} = \frac{w_2 (a_{11}(\mathbf{w}) a_{22}(\mathbf{w}) - a_{12}(\mathbf{w}) a_{21}(\mathbf{w}))}{(p_2)^2} \text{ and}$$
$$\frac{\partial p_1'(\mathbf{w})}{\partial w_2} = \frac{a_{21}(\mathbf{w}) p_2 - p_1 a_{22}(\mathbf{w})}{(p_2)^2} = \frac{w_1 (a_{12}(\mathbf{w}) a_{21}(\mathbf{w}) - a_{11}(\mathbf{w}) a_{22}(\mathbf{w}))}{(p_2)^2},$$

we obtain:

$$\forall \mathbf{w} \geq \mathbf{0},\ \frac{\partial p_1'(\mathbf{w})}{\partial w_1} \triangle\, w - \frac{\partial p_1'(\mathbf{w})}{\partial w_2} \triangle\, w > 0 \Leftrightarrow \forall \mathbf{w} \geq \mathbf{0},\ a_{11}(\mathbf{w}) a_{22}(\mathbf{w}) - a_{12}(\mathbf{w}) a_{21}(\mathbf{w}) > 0 \text{ and}$$
$$\forall \mathbf{w} \geq \mathbf{0},\ \frac{\partial p_1'(\mathbf{w})}{\partial w_1} \triangle\, w - \frac{\partial p_1'(\mathbf{w})}{\partial w_2} \triangle\, w < 0 \Leftrightarrow \forall \mathbf{w} \geq \mathbf{0},\ a_{11}(\mathbf{w}) a_{22}(\mathbf{w}) - a_{12}(\mathbf{w}) a_{21}(\mathbf{w}) < 0.$$

From (7), we obtain:

$$\forall \mathbf{w} \geq \mathbf{0},\ \frac{\partial p_1'(\mathbf{w})}{\partial w_1} \triangle\, w - \frac{\partial p_1'(\mathbf{w})}{\partial w_2} \triangle\, w > 0; \text{ or } \forall \mathbf{w} \geq \mathbf{0},\ \frac{\partial p_1'(\mathbf{w})}{\partial w_1} \triangle\, w - \frac{\partial p_1'(\mathbf{w})}{\partial w_2} \triangle\, w < 0.$$

This implies that $p_1'(\mathbf{w})$ is monotonic in relation to the change in factor prices $\mathbf{w} \to (w_1 + \triangle\, w, w_2 - \triangle\, w)$. In other words, $p_1'(\mathbf{w})$ and $\mathbf{w}$ are global univalent. This means that the FPET holds under condition (7). ■

7.2 The Proof of Theorem 2

First, we shall prove the following lemma.

Lemma 1: Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a continuous and partially differentiable mapping, where $f_{ij} \equiv \frac{\partial f_j}{\partial x_i}$. Suppose that there are $2n$ positive numbers m_k, M_k, for $k = 1, 2 \cdots, n$, such that the absolute values of the successive principal minors satisfy:

$$m_k \leqq \left| \det \begin{bmatrix} f_{11} & \cdots & f_{1k} \\ \vdots & \ddots & \vdots \\ f_{k1} & \cdots & f_{kk} \end{bmatrix} \right| \leqq M_k, \text{ for } k = 1, 2, \cdots, n,$$

in the whole of $\mathbb{R}^n$. Then, the system of equations, $f(\mathbf{x}) = \mathbf{a}$, has a unique solution in $\mathbb{R}^n$ for any given vector, $\mathbf{a} > \mathbf{0}$.

Proof: See Nikaido (1972). ■

In order to prove Theorem 2, it is useful to transform the variables in the cost function as follows: $\omega \equiv \ln \mathbf{w} = [\ln w_i]$ and $\pi \equiv \ln \mathbf{p} = [\ln p_i]$. Therefore, $\pi = \ln c(e^{\omega}) \equiv \varphi(\omega)$. Function φ is continuous and differentiable in $\mathbb{R}^n$. Therefore, $\frac{\partial \ln p_j}{\partial \ln w_i} = \frac{w_i}{c_j} \frac{\partial c_j}{\partial w_i} = \frac{w_i}{c_j} \times c_{ij}(\mathbf{w}) = \frac{c_{ij}(\mathbf{w}) w_i}{p_j} = \alpha_{ij}$ holds. In other words,

$\frac{\partial \ln p_j}{\partial \ln w_i}$ is the share of factor i's growth rate relative to the growth rate of the production cost of commodity j. Therefore, matrix $\widetilde{\mathbf{A}}$ is the Jacobian matrix of $\pi = \varphi(\omega)$. As such, we can prove Theorem 2.

Proof: The system of equations $f(\mathbf{x}) = \mathbf{a}$ in Lemma 1 corresponds to $\varphi(\omega) = \pi$. Because of $f_{ij} = \alpha_{ij} \geqq 0$, δ_k in (10) can serve as m_k in the Lemma. As previously mentioned, $\widetilde{\mathbf{A}}$ is a stochastic matrix, which implies that $\alpha_{ij} \in [0,1]$ and that $\sum_{i=1}^{n} \alpha_{ij} = 1$. Since the determinant is a polynomial of its elements, it is clear that the principal minors are bounded from above, which justifies the existence of M_k. Therefore, the system of equations $\varphi(\omega) = \pi$ satisfies Lemma 1. This immediately shows that $\varphi(\omega) = \pi$ has a unique solution for any given vector, $\pi > \mathbf{0}$. Therefore, $c(\mathbf{w}) = \mathbf{p}$ has a unique solution for any given vector, $\mathbf{p} > \mathbf{0}$.

7.3 The Proof of Theorem 4

Proof: Here, Function $\pi(\omega) = \ln c(e^{\omega})$ from Theorem 2 is used. From Assumption 2.4.2, we know that $\pi : \mathbb{R}^n \to \mathbb{R}^n$ and $J\pi(\omega) = \widetilde{\mathbf{A}} = [\alpha_{ij}]$, where $J\pi(\omega)$ is the Jacobian matrix of $\pi(\omega)$. Since, as previously mentioned, $\widetilde{\mathbf{A}}$ is a stochastic matrix (which implies $\alpha_{ij} \in [0,1]$), the absolute values of all elements of $J\pi(\omega)$ are uniformly bounded. Furthermore, we assume $\left|\det \widetilde{\mathbf{A}}\right| > \varepsilon > 0$. Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a continuous and differentiable function. The sufficient condition for $\left\|(JF(\mathbf{x}))^{-1}\right\| \leqq k$ is that i) $|JF(\mathbf{x})| \geqq \varepsilon > 0$, and ii) the absolute values of all elements of $JF(\mathbf{x})$ must be uniformly bounded, where $\|\|$ is a norm defined in $\mathbb{R}^n$ as $\|T\| \equiv \max_{\|\mathbf{x}\|=1} \|T\mathbf{x}\|$. Moreover, if $\det JF(\mathbf{x}) \neq 0$ and $\left\|(JF(\mathbf{x}))^{-1}\right\| \leqq k$ for some $k > 0$, then F will be a homeomorphism (Berger, 1977, p. 222). Therefore, $\pi : \mathbb{R}^n \to \mathbb{R}^n$ is a homeomorphism, which means that $c : \mathbb{R}^n_{++} \to \mathbb{R}^n_{++}$ is also a homeomorphism. ■

7.4 The Proof of Theorem 5

First, we shall present some lemmas necessary for the proof.

Lemma 2: Let $F : D \subset \mathbb{R}^k \to \mathbb{R}$ be a concave function and $\beta \in \partial F(\mathbf{X}_0)$ and $\beta \in \partial F(\mathbf{X}_1)$. Then, for $\forall \mu \in [0,1]$, $\beta \in \partial F(\mathbf{X}_\mu)$ where $\mathbf{X}_\mu = \mu \mathbf{X}_0 + (1-\mu)\mathbf{X}_1$. Furthermore, for $\forall \mu \in [0,1]$, $F(\mathbf{X}_\mu) = F(\mathbf{X}_0) + \beta(\mathbf{X}_\mu - \mathbf{X}_0)$ so that $\{\beta(\mathbf{X}_\mu - \mathbf{X}_0), (\mathbf{X}_\mu - \mathbf{X}_0)\}$ is a linear segment at $\mathbf{X}_0$.

Proof: Since β is the sub-gradient of F at $\mathbf{X}_0$ and $\mathbf{X}_1$, the following inequalities hold for all $\mathbf{X} \in D$:

$$F(\mathbf{X}) \leqq F(\mathbf{X}_0) + \beta(\mathbf{X} - \mathbf{X}_0), \quad F(\mathbf{X}) \leqq F(\mathbf{X}_1) + \beta(\mathbf{X}_\mu - \mathbf{X}_1).$$

Multiplying the inequalities by μ and $1-\mu$, respectively, yields:

$$F(\mathbf{X}) \leqq \mu F(\mathbf{X}_0) + (1-\mu) F(\mathbf{X}_1) + \beta(\mathbf{X} - \mathbf{X}_\mu), \tag{36}$$

for $\forall \mu \in [0,1]$. On the other hand, the concavity of F ensures that

$$F(\mathbf{X}_\mu) \geqq \mu F(\mathbf{X}_0) + (1-\mu) F(\mathbf{X}_1). \tag{37}$$

From (36) and (37), we know that $F(\mathbf{X}) \leqq F(\mathbf{X}_\mu) + \beta(\mathbf{X} - \mathbf{X}_\mu)$, which implies that $\beta \in \partial F(\mathbf{X}_\mu)$. Consequently, we obtain:

$$F(\mathbf{X}_0) \leqq F(\mathbf{X}_\mu) + \beta(\mathbf{X}_0 - \mathbf{X}_\mu) \text{ and } F(\mathbf{X}_\mu) \leqq F(\mathbf{X}_0) + \beta(\mathbf{X}_\mu - \mathbf{X}_0),$$

from which $F(\mathbf{X}_\mu) = F(\mathbf{X}_0) + \beta(\mathbf{X}_\mu - \mathbf{X}_0)$. Therefore, $\{\beta(\mathbf{X}_\mu - \mathbf{X}_0), (\mathbf{X}_\mu - \mathbf{X}_0)\}$ is a linear segment at $\mathbf{X}_0$. ■

Lemma 3: Factor prices equalise for $\mathbf{p} \in \Pi, \mathbf{V}^c \in \Gamma^c(\mathbf{p})$ if and only if

$$R^c(\mathbf{p}, \mathbf{V}^c) = R_0^c(\mathbf{p}) + \sum_{i=1}^{N} R_i(\mathbf{p}) V_i^c, \tag{38}$$

where $\mathbf{V}^c \equiv [V_i^c]$ for $c = 1, \cdots, C$. In this case, R^c is differentiable so that

$$\mathbf{W} = \nabla_{\mathbf{V}} R^c(\mathbf{p}, \mathbf{V}^c) = [R_1(\mathbf{p}), \cdots, R_N(\mathbf{p})]. \tag{39}$$

for all countries.

Proof: *Necessity.* If the profit function takes the form presented in (38), then it is differentiable and (39) holds, which is consistent with (24).

Sufficiency. From Definition 2.5.1, $(\Gamma_N^c(\mathbf{p}))_{c=1,\cdots,C}$ is defined for $\mathbf{p} \in \Pi$ where $\Pi \subseteq \mathbb{R}_+^M$ is a non-empty and open convex set. Moreover, $\mathbf{W} \in \mathbb{R}_+^N$ exists and $\mathbf{W} = \nabla R^c(\mathbf{p}, \mathbf{V}^c)$, for $\forall c = 1, \cdots, C$, for an arbitrary profile $(\mathbf{V}^c(\mathbf{p}))_{c=1,\cdots,C} \in \underset{c=1,\cdots,C}{\times} \Gamma_N^c(\mathbf{p})$. For $c = 1, \cdots, C, i = 1, \cdots, N$, and $\mathbf{V}^c(\mathbf{p}, \mathbf{V}^c) \in \Gamma_N^c(\mathbf{p})$,

$$W_i = \lim_{\delta_i^c \to 0} \frac{R^c(\mathbf{p}, \mathbf{V}^c + (\delta_i^c, \mathbf{0}_{-i})) - R^c(\mathbf{p}, \mathbf{V}^c)}{(V_i^c + \delta_i^c) - V_i^c}$$

holds, where $\mathbf{0}_{-i}$ stands for the vector of order $N-1$ excluding the ith element. Since, from Definition 2.5.1, the assumption for Lemma 2 is satisfied by $\mathbf{W}$ as shown above, the following is satisfied for arbitrary $\mathbf{V}^c, \overline{\mathbf{V}}^c \in \Gamma_N^c(\mathbf{p})$:

$$R^c(\mathbf{p}, \mathbf{V}^c) = R^c\left(\mathbf{p}, \overline{\mathbf{V}}^c\right) + \mathbf{W}\left(\mathbf{V}^c - \overline{\mathbf{V}}^c\right).$$

Since $\mathbf{W}$ is independent of $\mathbf{V}^c$, $R_i : \Pi \to \mathbb{R}_+$ exists for $i = 1, \cdots, N$. Since $W_i = R_i(\mathbf{p})$, we obtain:

$$R^c(\mathbf{p}, \mathbf{V}^c) = R^c\left(\mathbf{p}, \overline{\mathbf{V}}^c\right) + \sum_{i=1}^{N} R_i(\mathbf{p})\left(V_i^c - \overline{V}_i^c\right), \text{ for } \forall c = 1, \cdots C.$$

By treating $\overline{\mathbf{V}}^c$ as the vector fixed in $\Gamma_N^c(\mathbf{p})$, we obtain: $R_0^c(\mathbf{p}) \equiv R^c\left(\mathbf{p}, \overline{\mathbf{V}}^c\right) - \sum_{i=1}^{N} R_i(\mathbf{p}) \overline{V}_i^c$. ■

Lemma 4: Suppose that $F : D \subseteq \mathbb{R}^k \to \mathbb{R}$ is concave and $(\beta\delta^1, \delta^1)$ and $(\beta\delta^2, \delta^2)$ at $\mathbf{X}$ are linear segments where $\beta \in \partial F(\mathbf{X})$ in $\mathbf{X} \in D$. Then, $(\beta\delta, \delta)$ is also a linear segment at $\mathbf{X}$ for $\delta = \mu\delta^1 + (1-\mu)\delta^2$ for all $\mu \in [0, 1]$.

Proof: Let us define $\mathbf{X}_1 = \mathbf{X} + \lambda_1\delta^1$ and $\mathbf{X}_2 = \mathbf{X} + \lambda_2\delta^2$ where $\lambda_1 \in (-\varepsilon_1, \varepsilon_1)$ and $\lambda_2 \in (-\varepsilon_2, \varepsilon_2)$. Then, $\beta \in \partial F(\mathbf{X}_1)$ and $\beta \in \partial F(\mathbf{X}_2)$ hold based on Definition 2.5.2. Since Lemma 2 can be applied, we find that $\beta \in \partial F(\mathbf{X}_\mu)$ holds, where $\mathbf{X}_\mu$ is the convex combination of $\mathbf{X}_1$ and $\mathbf{X}_2$. Thanks to Lemma 2 $F(\mathbf{X}_\mu) = F(\mathbf{X}) + \beta(\mathbf{X}_\mu - \mathbf{X}_0)$ holds, which means that $(\beta\delta, \delta)$ is a linear segment of F in $\mathbf{X}$. ■

Lemma 5: Let R^c be the profit function defined by (23). A linear segment of R^c in $\mathbf{V}^c$ at $(\mathbf{p}, \mathbf{V}_0^c)$ is equivalent to a linear segment of G^c in $(\mathbf{Z}^c, \mathbf{V}^c)$ at $(\mathbf{Z}_0^c, \mathbf{V}_0^c)$, where $\mathbf{Z}_0^c$ is the profit maximiser in (23) given $(\mathbf{p}, \mathbf{V}_0^c)$ for some $\mathbf{p} \in \Pi$.

Proof: If R^c has a linear segment $(\mathbf{W}\delta, \delta)$ at $(\mathbf{p}, \mathbf{V}_0^c)$, then $\mathbf{W} = \nabla_{\mathbf{V}} R^c(\mathbf{p}, \mathbf{V}_0^c + \lambda\delta)$ for all $\lambda \in (-\varepsilon, \varepsilon)$ for some $\varepsilon > 0$. Letting $\mathbf{p} = (\mathbf{p}^y, \mathbf{p}^z)$,

$$\nabla_{\mathbf{V}} R^c(\mathbf{p}, \mathbf{V}_0^c) = \sum_{i=1}^{N} p_i^y \frac{\partial G^c(\mathbf{Z}_0^c, \mathbf{V}_0^c)}{\partial V_i} = \mathbf{p}^y \nabla_{\mathbf{V}} G^c(\mathbf{Z}_0^c, \mathbf{V}_0^c).$$

From the two equations shown above, we obtain:

$$
\begin{aligned}
\mathbf{W} = \nabla_{\mathbf{V}} R^c\left(\mathbf{p}, \mathbf{V}_0^c + \lambda\delta\right) &= \sum_{i=1}^{N} p_i^y \frac{\partial G^c\left(\mathbf{Z}^c\left(\lambda\right), \mathbf{V}_0^c + \lambda\delta\right)}{\partial V_i}, \\
&= \mathbf{p}^y \nabla_{\mathbf{V}} G^c\left(\mathbf{Z}^c\left(\lambda\right), \mathbf{V}_0^c + \lambda\delta\right),
\end{aligned}
$$

where $\mathbf{Z}^c\left(\lambda\right)$ is the profit maximiser at $\left(\mathbf{p}, \mathbf{V}_0 + \lambda\boldsymbol{\delta}\right)$. Let $\left(\mathbf{Z}_1^c, \mathbf{V}_1^c\right) \equiv \left(\mathbf{Z}^c\left(\lambda\right), \mathbf{V}_0 + \lambda\delta\right)$ for all $\lambda \in \left(-\varepsilon, \varepsilon\right)$. Since

$$
\mathbf{p}^y \nabla G^c\left(\mathbf{Z}^c, \mathbf{V}^c\right) = \left(\left(p_j^y \frac{\partial G^c\left(\mathbf{Z}^c, \mathbf{V}^c\right)}{\partial Z_j}\right)_{j=1,\cdots,M}, \left(p_i^y \frac{\partial G^c\left(\mathbf{Z}^c, \mathbf{V}^c\right)}{\partial V_j}\right)_{i=1,\cdots,N}\right)
$$

and

$$
\frac{\partial R^c}{\partial Z_j} = p_j^y \frac{\partial G^c\left(\mathbf{Z}^c, \mathbf{V}^c\right)}{\partial Z_j} + p_j^z = 0 \;\; \left(\forall j = 1, \cdots M\right),
$$

as such, we obtain:

$$
\mathbf{p}^y \nabla_{\mathbf{Z}} G^c\left(\mathbf{Z}^c, \mathbf{V}^c\right) = \left(p_j^y \frac{\partial G^c\left(\mathbf{Z}^c, \mathbf{V}^c\right)}{\partial Z_j}\right)_{j=1,\cdots,M} = -\mathbf{p}^z.
$$

Therefore, $\mathbf{p}^y \nabla G^c\left(\mathbf{Z}^c, \mathbf{V}_0^c\right) = \left(-\mathbf{p}^z, \mathbf{W}\right)$ holds. Similarly, $\mathbf{p}^y \nabla G^c\left(\mathbf{Z}^c\left(\lambda\right), \mathbf{V}_0^c + \lambda\delta\right) = \left(-\mathbf{p}^z, \mathbf{W}\right)$ also holds. Consequently, we obtain:

$$
\left(-\mathbf{p}^z, \mathbf{W}\right) = \mathbf{p}^y \nabla G^c\left(\mathbf{Z}_0^c, \mathbf{V}_0^c\right) = \mathbf{p}^y \nabla G^c\left(\mathbf{Z}_1^c, \mathbf{V}_1^c\right).
$$

From Lemma 2, therefore, we obtain: $\left(-\mathbf{p}^z, \mathbf{W}\right) = \mathbf{p}^y \nabla G^c\left(\mathbf{Z}_\mu^c, \mathbf{V}_\mu^c\right)$, where $\mathbf{Z}_\mu^c = \mu \mathbf{Z}_0^c + \left(1-\mu\right)\mathbf{Z}_1^c, \mathbf{V}_\mu^c = \mu \mathbf{V}_0^c + \left(1-\mu\right)\mathbf{V}_1^c$ for $\forall \mu \in [0,1]$. Here, letting $\beta = \left(-\mathbf{p}^z/\mathbf{p}^y, \mathbf{W}/\mathbf{p}^y\right)$ and $\delta = \left(\mathbf{Z}_1^c, \mathbf{V}_1^c\right) - \left(\mathbf{Z}_0^c, \mathbf{V}_0^c\right)$,we see that $\left(\boldsymbol{\beta}\delta, \delta\right)$ is a linear segment of G^c at $\left(\mathbf{Z}_0^c, \mathbf{V}_0^c\right)$.

In order to prove the converse, suppose that $\left(\psi^y, \psi^z, \delta\right)$ is a linear segment of G^c at $\left(\mathbf{Z}_0^c, \mathbf{V}_0^c\right)$. As such, the following holds:

$$
G^c\left(\mathbf{Z}_0^c, \mathbf{V}_0^c\right) + \lambda\psi^y = G^c\left(\mathbf{Z}_0^c + \lambda\boldsymbol{\psi}^y, \mathbf{V}_0^c + \lambda\delta\right).
$$

Therefore, the following relationships must hold for all $\lambda \in \left(-\varepsilon, \varepsilon\right)$ for some $\varepsilon > 0$:

$$
\mathbf{p}^y \nabla_{\mathbf{Z}} G^c\left(\mathbf{Z}_0^c + \lambda\psi^x, \mathbf{V}_0^c + \lambda\delta\right) = \mathbf{p}^y \nabla_{\mathbf{Z}} G^c\left(\mathbf{Z}_0^c, \mathbf{V}_0^c\right) = -\mathbf{p}^z,
$$

and

$$
\mathbf{p}^y \nabla_{\mathbf{V}} G^c\left(\mathbf{Z}_0^c + \lambda\psi^x, \mathbf{V}_0^c + \lambda\delta\right) = \mathbf{p}^y \nabla_{\mathbf{V}} G^c\left(\mathbf{Z}_0^c, \mathbf{V}_0^c\right) = \mathbf{W}.
$$

Because of $\nabla_{\mathbf{V}} R^c\left(\mathbf{p}, \mathbf{V}_0^c\right) = \mathbf{p}^y \nabla_{\mathbf{V}} G^c\left(\mathbf{Z}_0^c, \mathbf{V}_0^c\right)$, $\mathbf{W}{=}\boldsymbol{\nabla}_{\mathbf{V}} R_c\left(\mathbf{p}, \mathbf{V}_0^c\right)$. Similarly, $\nabla_{\mathbf{V}} R^c\left(\mathbf{p}, \mathbf{V}_0^c + \lambda\delta\right) = \mathbf{p}^y \nabla_{\mathbf{V}} G^c\left(\mathbf{Z}_0^c + \lambda\psi^c, \mathbf{V}_0^c + \lambda\delta\right)$ holds for ψ^z, thus satisfying $\mathbf{p}^y \psi^y = \mathbf{W}\delta {-} \mathbf{p}^z \psi^z$. Therefore, $\mathbf{W} = \boldsymbol{\nabla}_{\mathbf{V}} R^c\left(\mathbf{p}, \mathbf{V}_0^c + \lambda\delta\right)$ holds for all $\forall\lambda \in \left(-\varepsilon, \varepsilon\right)$. This means that $\left(\mathbf{W}\delta, \delta\right)$ is a linear segment of R^c in $\mathbf{V}$ at $\left(\mathbf{p}, \mathbf{V}_0^c\right)$. ■

Now, let us proceed to the proof of Theorem 5.

Proof: *Necessity.* Assume that the factor prices are equalised. As described in Lemma 3, the profit function, R^c, has N linear segments at any point $\left(\mathbf{p}, \mathbf{V}^c\right) \in \Pi \times \Gamma_N^c\left(\mathbf{p}\right)$. Let us denote these linear segments as $\left(R_i\left(\mathbf{p}\right), \delta_i\right)$, where δ_i for $i = 1, \cdots, N$ can be chosen as the standard basis vector for $\mathbb{R}^N$. Furthermore, let $\mathbf{Z}^{c*}$ be the equilibrium production vector given $\left(\mathbf{p}, \mathbf{V}_0^c\right)$. From Lemma 5, we see that if $\left(R_i\left(\mathbf{p}\right), \delta_i\right)$ is a linear segment for R^c at $\left(\mathbf{p}, \mathbf{V}^c\right)$ there exists a vector, $\left(\psi_i^c, \delta_i\right)$, that is a linear segment of G^c at $\left(\mathbf{Z}^{c*}, \mathbf{V}^c\right)$. Therefore, at any point $\left(\mathbf{p}, \mathbf{V}^c\right) \in \Pi \times \Gamma_N^c\left(\mathbf{p}\right)$, there exist N linear segments $\left(\psi_i^c\left(\mathbf{p}, \mathbf{V}^c\right), \delta_i\right)$ for G^c at $\left(\mathbf{Z}^{c*}, \mathbf{V}_0^c\right)$. Since $\left(\psi_i, \delta_i\right)$ are linear segments of G^c, they are directions of linearity of T^c.

Since δ_i for $i = 1, \cdots, N$ are the standard basis vectors, the following holds for $\lambda \in (-\varepsilon, \varepsilon)$ from Lemma 3:

$$R^c \left(\mathbf{p}, \mathbf{V}^c + \lambda \left(\delta_i, \mathbf{0}_{-i}\right)\right) = R^c \left(\mathbf{p}\right) + \lambda R_i \left(\mathbf{p}\right).$$

Moreover, from the definitions of the profit function and linear segments, we can see that:

$$\mathbf{p} \left\{\mathbf{X}^{c*} + \lambda \psi_i^c \left(\mathbf{p}, \mathbf{V}^c\right)\right\} = R^c \left(\mathbf{p}\right) + \lambda R_i \left(\mathbf{p}\right).$$

Since $\mathbf{p}\mathbf{X}^{c*} = R^c(\mathbf{p})$ if $\lambda = 0$, we obtain:

$$\mathbf{p}\psi_i^c \left(\mathbf{p}, \mathbf{V}^c\right) = R_i \left(\mathbf{p}\right). \tag{40}$$

(40) must be satisfied for $\forall \mathbf{p} \in \Pi$. This implies that $\psi_i^c(\mathbf{p}, \mathbf{V}^c)$ is independent of $\mathbf{V}^c \in \Gamma_N^c(\mathbf{p})$ and is the same for all countries.

Sufficiency. We assume that production function G^c is N linear segments $(\psi_i(\mathbf{p}), \delta_i)$ for $i = 1, \cdots, N$, where δ_i are standard basis vectors. From Lemma 5, R^c has N linear segments in $\mathbf{V}$ given $\mathbf{p}$. For $i = 1, \cdots, N$, therefore, $R^c(\mathbf{p}, \mathbf{V}^c + \lambda(\delta_i, \mathbf{0}_{-i})) = R^c(\mathbf{p}) + \lambda R_i(\mathbf{p})$ holds for $\forall \lambda \in (-\varepsilon, \varepsilon)$ for some $\varepsilon > 0$. Based on Lemma 4, it is clear that $\left(\frac{1}{N} R(\mathbf{p}), \frac{\delta}{N}\right)$ is a linear segment of R^c in $\mathbf{V}$ at $(\mathbf{p}, \mathbf{V}^c)$. In other words, the following holds for $\forall \lambda \in (-\varepsilon/N, \varepsilon/N)$ for some $\varepsilon/N > 0$:

$$R^c \left(\mathbf{p}, \mathbf{V}^c + \lambda \delta\right) = R^c \left(\mathbf{p}, \mathbf{V}^c\right) + \lambda \sum_{i=1}^{N} R_i \left(\mathbf{p}\right) \delta_i.$$

By applying Lemma 2 and using the same logic as that used in the proof of Lemma 3, we obtain what follows:

$$R^c \left(\mathbf{p}, \mathbf{V}^c\right) = R_0^c \left(\mathbf{p}\right) + \sum_{i=1}^{N} R_i^c \left(\mathbf{p}\right) V_i^c.$$

Here, ψ_i:$\Pi \to \mathbb{R}$, for $i = 1, \cdots, N$, are common to all countries and, as shown in (40), $\mathbf{p}\psi_i(\mathbf{p}) = R_i^c(\mathbf{p}), c = 1, \cdots, C$, holds for $i = 1, \cdots, N$. In other words, we obtain:

$$R^c \left(\mathbf{p}, \mathbf{V}^c\right) = R_0^c \left(\mathbf{p}\right) + \sum_{i=1}^{N} R_i \left(\mathbf{p}\right) V_i^c, \text{ for } \forall c = 1, \cdots, C.$$

From Lemma (3), we can see that the factor prices are equalised for arbitrary $\left(\mathbf{p}, \left(\mathbf{V}^c\right)_{c=1,\cdots,C}\right) \in \Pi \times \left(\underset{c=1,\cdots,C}{\times} \Gamma_N^c\right)$. ■

7.5 The Proof of Theorem 6

Proof: *Necessity.* We assume that factor prices are equalised. Theorem 5 shows that there exist N vectors, $(\psi_i(\mathbf{p}), \delta_i(\mathbf{p}))$, for $i = 1, \cdots, N$, that are directions of linearity for T^c. In other words, there exists $\varepsilon > 0$ such that for $\lambda \in (-\varepsilon, \varepsilon)$:

$$T^c \left(\mathbf{X}^{c*} + \lambda \psi_i \left(\mathbf{p}\right), \mathbf{V}^c + \lambda \delta_i\right) = 0,$$

differentiating which with respect to λ yields:

$$\nabla_{\mathbf{X}} T^c \left(\mathbf{X}^{c*}, \mathbf{V}^c\right) \mathbf{\Psi} + \nabla_{\mathbf{V}} T^c \left(\mathbf{X}^{c*}, \mathbf{V}^c\right) \mathbf{\Omega} = \mathbf{0}.$$

This is simply a repetition of (28). From the above, we obtain:

$$\begin{aligned}
\nabla_{\mathbf{XX}} T^c \left(\mathbf{X}^{c*}, \mathbf{V}^c\right) \mathbf{\Psi} + \nabla_{\mathbf{XV}} T^c \left(\mathbf{X}^{c*}, \mathbf{V}^c\right) \mathbf{\Omega} &= \mathbf{0}, \\
\nabla_{\mathbf{XV}} T^c \left(\mathbf{X}^{c*}, \mathbf{V}^c\right) \mathbf{\Psi} + \nabla_{\mathbf{VV}} T^c \left(\mathbf{X}^{c*}, \mathbf{V}^c\right) \mathbf{\Omega} &= \mathbf{0}.
\end{aligned}$$

The former is (26) and pre-multiplying it by $\mathbf{\Psi}^T$, yields:

$$\mathbf{\Psi}^T \nabla_{\mathbf{XX}} T^c \left(\mathbf{X}^{c*}, \mathbf{V}^c\right) \mathbf{\Psi} + \mathbf{\Psi}^T \nabla_{\mathbf{XV}} T^c \left(\mathbf{X}^{c*}, \mathbf{V}^c\right) \mathbf{\Omega} = \mathbf{0}. \tag{41}$$

Pre-multiplying the latter by $\mathbf{\Omega}^T$ yields:

$$\mathbf{\Omega}^T \nabla_{\mathbf{XV}} T^c \left(\mathbf{X}^{c*}, \mathbf{V}^c\right) \mathbf{\Psi} + \mathbf{\Omega}^T \nabla_{\mathbf{VV}} T^c \left(\mathbf{X}^{c*}, \mathbf{V}^c\right) \mathbf{\Omega} = \mathbf{0},$$

transposing this yields:

$$\mathbf{\Psi}^T \nabla_{\mathbf{XV}} T^c \left(\mathbf{X}^{c*}, \mathbf{V}^c\right) \mathbf{\Omega} + \mathbf{\Omega}^T \nabla_{\mathbf{VV}} T^c \left(\mathbf{X}^{c*}, \mathbf{V}^c\right) \mathbf{\Omega} = \mathbf{0}. \tag{42}$$

Since $\mathbf{\Omega}$ is an identity matrix, $\nabla_{\mathbf{VV}} T^c \left(\mathbf{X}^{c*}, \mathbf{V}^c\right) = \mathbf{\Psi}^T \nabla_{\mathbf{XX}} T^c \left(\mathbf{X}^{c*}, \mathbf{V}^c\right) \mathbf{\Psi}$ is obtained from (41) and (42).

Sufficiency. (26) and (27) imply the following relationships:

$$\begin{bmatrix} \nabla_{\mathbf{XX}} T^c \left(\mathbf{X}^{c*}, \mathbf{V}^c\right) & \nabla_{\mathbf{XV}} T^c \left(\mathbf{X}^{c*}, \mathbf{V}^c\right) \end{bmatrix} \begin{bmatrix} \psi_i \left(\mathbf{p}\right) \\ \delta_i \left(\mathbf{p}\right) \end{bmatrix} = \mathbf{0},$$

$$\begin{bmatrix} \nabla_{\mathbf{XV}} T^c \left(\mathbf{X}^{c*}, \mathbf{V}^c\right) & \nabla_{\mathbf{VV}} T^c \left(\mathbf{X}^{c*}, \mathbf{V}^c\right) \end{bmatrix} \begin{bmatrix} \psi_i \left(\mathbf{p}\right) \\ \delta_i \left(\mathbf{p}\right) \end{bmatrix} = \mathbf{0}.$$

These equations imply that the Hessian matrix of T^c has N eigenvectors that satisfy (28) and that are associated with zero eigenvalues. The eigenvectors are given by $\left(\psi_i \left(\mathbf{p}\right), \delta_i \left(\mathbf{p}\right)\right)$, $i = 1, \cdots, N$. These eigenvectors are directions of linearity of T^c. Therefore, Theorem 5 implies that the FPET holds. ■

7.6 The Proof of Theorem 7

Proof: The wage-profit curve in the case of $n = 2$ is given as:

$$w^1 \left(r\right) = \frac{\left\{1 - \left(1+r\right) a_{11} \left(\mathbf{p}, w, r\right)\right\} \left\{1 - \left(1+r\right) a_{22} \left(\mathbf{p}, w, r\right)\right\} - \left(1+r\right)^2 a_{12} \left(\mathbf{p}, w, r\right) a_{21} \left(\mathbf{p}, w, r\right)}{l_1 \left\{1 - \left(1+r\right) a_{22} \left(\mathbf{p}, w, r\right)\right\} + \left(1+r\right) l_2 \left(\mathbf{p}, w, r\right) a_{21} \left(\mathbf{p}, w, r\right)},$$

where $w^1 \left(r\right) \equiv \frac{w(r)}{p_1}$. The relative price of commodity 2 is given by:

$$p \left(r\right) = \frac{l_2 \left(\mathbf{p}, w, r\right) \left\{1 - \left(1+r\right) a_{11} \left(\mathbf{p}, w, r\right)\right\} + \left(1+r\right) l_1 \left(\mathbf{p}, w, r\right) a_{12} \left(\mathbf{p}, w, r\right)}{l_1 \left(\mathbf{p}, w, r\right) \left\{1 - \left(1+r\right) a_{22} \left(\mathbf{p}, w, r\right)\right\} + \left(1+r\right) l_2 \left(\mathbf{p}, w, r\right) a_{21} \left(\mathbf{p}, w, r\right)}.$$

Therefore, we obtain:

$$\frac{\mathrm{d}p \left(r\right)}{\mathrm{d}r} = \frac{l_1 \left(l_1 a_{12} + l_2 a_{22}\right) - l_2 \left(l_1 a_{11} + l_2 a_{21}\right)}{l_1 \left\{1 - \left(1+r\right) a_{22}\right\} + \left(1+r\right) l_2 a_{21}}.$$

If the techniques are productive, then the denominator is positive for all feasible rates of profit. Therefore, the sign of $\frac{\mathrm{d}p}{\mathrm{d}r}$ is solely dependent on the numerator. The relative price and the rate of profit have the following relationship if the techniques are productive:

$$\frac{\mathrm{d}p}{\mathrm{d}r} \lesseqqgtr 0 \Leftrightarrow \frac{a_{12} \left(\mathbf{p}, w, r\right) + p a_{22} \left(\mathbf{p}, w, r\right)}{l_2 \left(\mathbf{p}, w, r\right)} \lesseqqgtr \frac{a_{11} \left(\mathbf{p}, w, r\right) + p a_{21} \left(\mathbf{p}, w, r\right)}{l_1 \left(\mathbf{p}, w, r\right)}.$$

Here, $\frac{a_{11}(\mathbf{p},w,r) + p a_{21}(\mathbf{p},w,r)}{l_1(\mathbf{p},w,r)}$ is the capital intensity of industry 1 and $\frac{a_{12}(\mathbf{p},w,r) + p a_{22}(\mathbf{p},w,r)}{l_2(\mathbf{p},w,r)}$ is that of industry 2. In other words, whether the relative price is monotonically increasing or decreasing with respect to the rate of profit is dependent on the relative size of the capital intensities. The relative price is a monotonically decreasing function with respect to the rate of profit if and only if industry 1 is more capital intensive than industry 2. Conversely, it is monotonically increasing if and only if industry 2 is more capital intensive than industry 1.[28] In other words, the relative price and the rate of profit have no monotonic relationship (i.e. global univalent) if and only if capital intensity reversal takes place at the threshold of the relative price. ■

7.7 The Proof of Theorem 8

Proof: Proving the theorem for the SSS-II condition will be sufficient. Since we assume that $-\bar{\mathbf{s}}\mathbf{B}_3 < 0$ holds and matrix $[\mathbf{B}_1 - (1+r)\,\mathbf{I}]$ has all positive diagonal elements and all negative off-diagonal elements, the solution for (33) is $\mathbf{p} > \mathbf{0}$ for $r \in [0, r^*)$ where r^* is the maximum rate of profit as determined by the Frobenius root of $\mathbf{B}_1$ (Takayama, 1985, p. 393). This implies that $[\mathbf{B}_1 - (1+r)\,\mathbf{I}]^{-1} < \mathbf{0}$. Differentiating (33) with respect to r, yields $\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}r}[\mathbf{B}_1 - (1+r)\,\mathbf{I}] - \mathbf{p} = \mathbf{0}$. In other words, we obtain:

$$\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}r} = \mathbf{p}\,[\mathbf{B}_1 - (1+r)\,\mathbf{I}]^{-1} < 0.$$

We can prove the theorem for the SSS-I condition in a similar manner. ■

7.8 The Proof of Theorem 9

Proof: Proving the theorem for the SSS-II condition will be sufficient. The first n equations of (34) are given as $\frac{\mathrm{d}\mathbf{q}}{\mathrm{d}r} = \mathbf{p} + (1+r)\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}r}$, which implies that:

$$\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}r}\mathbf{B}_1 + \frac{\mathrm{d}\bar{\mathbf{s}}}{\mathrm{d}r}\mathbf{B}_3 = \mathbf{p} + (1+r)\,\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}r}.$$

Since consumption prices are exogenously given, $\frac{\mathrm{d}\bar{\mathbf{s}}}{\mathrm{d}r} = \mathbf{0}$. Consequently, we obtain:

$$\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}r}\,[\mathbf{B}_1 - (1+r)\,\mathbf{I}] = \mathbf{p}.$$

If the SSS-II condition is satisfied, then $[\mathbf{B}_1 - (1+r)\,\mathbf{I}]^{-1} < \mathbf{0}$. Therefore, we obtain:

$$\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}r} = \mathbf{p}\,[\mathbf{B}_1 - (1+r)\,\mathbf{I}]^{-1} < \mathbf{0}.$$

We can prove the theorem for the SSS-I condition in a similar manner. ■

[28] Whether the relative price is monotonically increasing or decreasing can be determined by the technical coefficients, independently of the price:

$$\frac{\mathrm{d}p}{\mathrm{d}r} \lesseqqgtr 0 \Longleftrightarrow \frac{l_1(\mathbf{p},w,r)\,a_{12}(\mathbf{p},w,r) + l_2(\mathbf{p},w,r)\,a_{22}(\mathbf{p},w,r)}{l_2(\mathbf{p},w,r)} \lesseqqgtr \frac{l_1(\mathbf{p},w,r)\,a_{11}(\mathbf{p},w,r) + l_2(\mathbf{p},w,r)\,a_{21}(\mathbf{p},w,r)}{l_1(\mathbf{p},w,r)}.$$

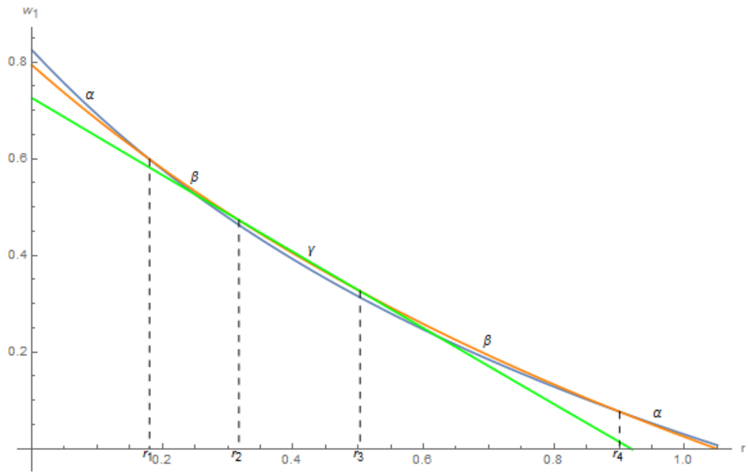

Figure 1: The wage-profit curve of sector 1

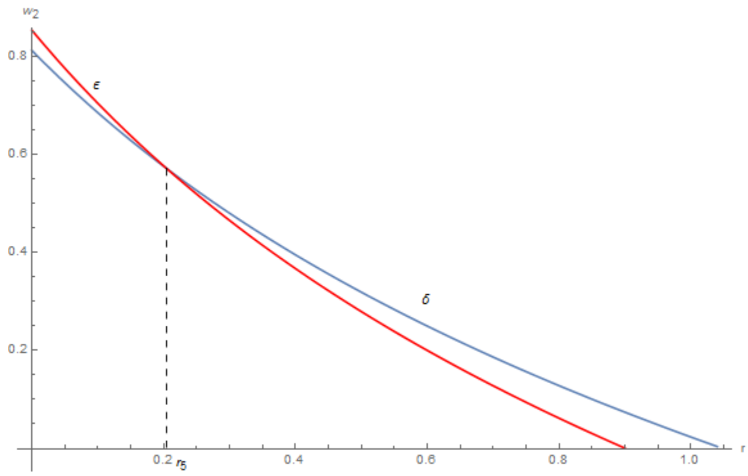

Figure 2: The wage-profit curve of sector 2

r	w_1	w_2	$\frac{p_2}{p_1}$	k_1	k_2
0	0.826	0.854	0.966	1.451	1.565
0.1	0.692	0.705	0.981	1.336	1.36
0.2	0.58	0.577	1.004	1.073	1.392
0.3	0.488	0.481	1.015	1.021	1.125
0.4	0.407	0.396	1.0295	0.797	1.075
0.5	0.328	0.319	1.0277	0.796	1.015
0.6	0.258	0.250	1.030	0.894	0.967
0.7	0.193	0.187	1.0320	0.859	0.924
0.8	0.133	0.128	1.0323	0.827	0.884
0.9	0.076	0.074	1.031	0.798	0.846
1.0	0.029	0.023	1.238	0.796	0.978
1.04	0.011	0.004	2.723	0.783	2.118

Table 1: The real wage rate, relative price, and capital intensity

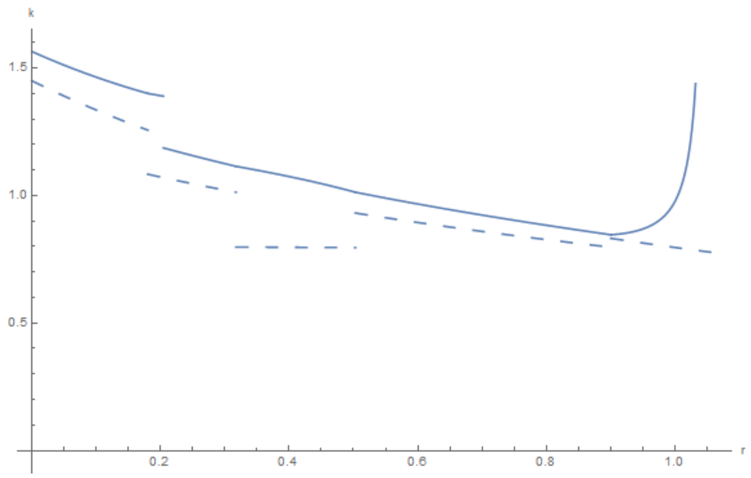

Figure 3: The capital intensity (real line shows for sector 2 and dashed line for sector 1)

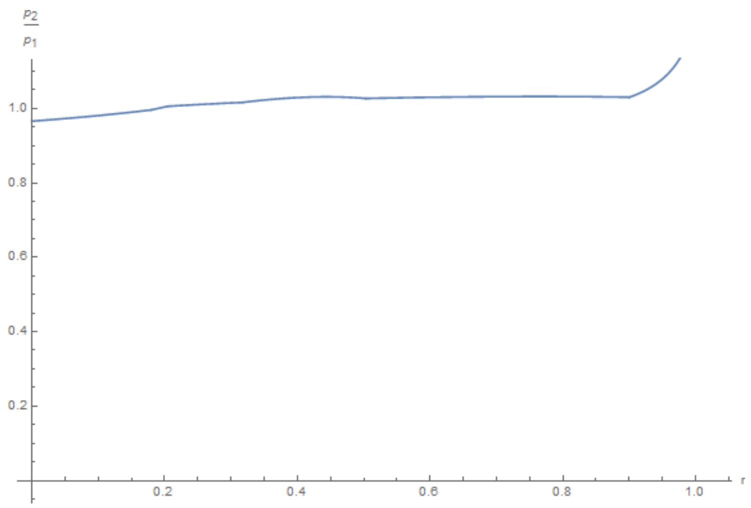

Figure 4: The relative price