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On the Ricardian Invariable Measure of Value in General Convex Economies:

Applicability of the Standard Commodity*

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Abstract

The purpose of this paper is to examine the critical arguments made by Burmeister, Samuelson, and others, with respect to Sraffa (1960). Sraffa did not address these arguments, but they are relevant from the viewpoint of modern economic theories. In his arguments about the standard commodity, Sraffa assumed that a change in income distribution has no effect on the output level and choice of techniques. However, modern economic theories allow interdependence among changes in income distribution, output level, and choice of techniques. Therefore, it is interesting to consider the existence of an invariable measure of value and linearity of income distribution in a model where such interdependence is discussed. We assume general convex economies with non-increasing returns to scale. In this model, we obtain the conditions under which the existence of an invariable measure of value and the validity of the linearity of income distribution are assured.

Keywords: Ricardo's invariable measure of value, Sraffa's standard commodity, General convex economies, Linear relation of income distribution

JEL Classifications: B51, D30, D51

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1   Introduction

As is well known, in his later years, David Ricardo intensively searched for an invariable measure of value.[1] His struggle to find it is shown in his *Principles*, his papers entitled 'Absolute Value and Exchangeable Value', which were written in the last few weeks of his life (Ricardo, 1951D, pp. 361–412), and a set of his letters to Malthus, McCulloch, and others.

An invariable measure of value can be defined as a measure that is invariable with respect to changes in both income distribution and technique (Ricardo, 1951A, chap. 1). Without an invariable measure of value, we cannot in general distinguish between changes in the price of the numéraire and that of the commodity measured, when relative prices change. The advantage of the invariable measure of value, if it exists, is that we can distinguish between the variations which belong to the commodity itself and those which are occasioned by a variation in the medium by which values or prices are expressed (Ricardo, 1951A, p. 48). For Ricardo, the pursuit of the invariable measure of value is directly related to the completion of the embodied labour theory of value,[2] although the importance of the measure of value was not adequately understood by his contemporaries such as Malthus.[3]

Even though it is true that the embodied labour theory of value cannot generally hold when the rate of profit is positive, it does not mean that the invariable measure of value is no longer significant. The purpose of Ricardo's construction of the invariable measure of value is to build a solid foundation not only to measure such important variables as national income or national wealth precisely, but also to compare those variables intertemporally. Therefore, no one can deny the importance of the invariable measure of value even today.

In the 20th century, Sraffa (1960) revived the concept of the invariable measure of

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[1]Ricardo's concern about an invariable measure of value appeared as early as in his contributions to the 'bullionist' controversy. As a bullionist, he wrote some notes on the stability of the general price level during the Napoleonic wars, which are collected in Ricardo (1951C). He had already pointed out the need for an invariable measure of value, which would enable an intertemporal comparison of values, and argued that such a measure did not exist in reality; however, money could be regarded as an invariable measure of value at least as the first approximation (see especially Ricardo, 1951C, p. 65). However, his arguments at this stage were not rigorously based on the theory of value. See Kurz and Salvadori (1993) concerning the conceptual transition of Ricardo's invariable measure of value.

[2]Ricardo (1952C, p. 358) said, 'As soon as we are in possession of the knowledge of the circumstances which determine the value of commodities, we are enabled to say what is necessary to give us an invariable measure of value.' See also Sraffa (1951) for the transition of Ricardo's theory of value in detail.

[3]See Porta (1992) concerning debates on the measure of value between Ricardo and Malthus.

value, which had fallen into oblivion since the so-called Marginal Revolution in the 1870s. Unlike Ricardo, he divided the problem of identifying an invariable measure of value into two parts: the first is to search for a measure of value that is invariable with respect to changes in technique, left aside the change in income distribution, and the other is to search for a measure of value that is invariable with respect to the change in income distribution, left aside the change in technique. Sraffa exclusively concentrated on the latter by constructing a special, composite commodity termed the *standard commodity*.[4] As we will see, he also demonstrated an interesting relationship with respect to income distribution if the standard commodity is adopted as the numéraire: the linear relationship of income distribution. Although many economists have paid great attention to the results obtained by Sraffa, it seems that they have not reached a consensus on evaluating Sraffa (1960). Some economists appreciate him, whereas others do not unconditionally admit the significance of the standard commodity and the linearity of income distribution. In particular, those who are critical of Sraffa (1960) regard the assumption of a fixed technique without constant returns to scale as being too restrictive, and thus, downgrade the relevance of Sraffa (1960). As we shall see, Burmeister (1968, 1975, 1977, 1980, 1984), Samuelson (2000, 2008), Samuelson and Etula (2006), and others claimed that Sraffa's analyses are irrelevant without the assumption of constant returns to scale.

We think that the views of Burmeister, Samuelson, and others are worth examining, because they point out relevant problems from the viewpoint of modern economic theory, which Sraffa had not addressed. As we will see, in his arguments about the standard commodity, Sraffa (1960) assumed that a change in income distribution has no effect on the output level and choice of techniques. Of course, such an assumption is just an analytical device to construct a model; it is plausible that the change in income distribution is related to changes in output level or choice of techniques in actual economies. In fact, almost all modern economic theories admit interdependence among changes in income distribution, output level, and choice of techniques, even though the logical consequences of such interdependence are different among theories. Even those who are favourable to Sraffa would not be able to deny this interdependence. Curiously enough, there is little literature on whether or not an invariable measure of value and linearity of income distribution can be obtained in

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[4]Pasinetti's *dynamic standard commodity* is one of the examples that pay attention to the former. According to Pasinetti (1981, p. 105, n. 13), the economic system is, so to speak, 'frozen' at a given technique where Sraffa constructed the standard commodity whose value is invariant with respect to the change in income distribution; whereas the economic system is 'frozen' at a given income distribution where Pasinetti constructed the dynamic standard commodity which always requires the same quantity of 'augmented' labour through time. Therefore, we cannot analyse a change in income distribution by using the dynamic standard commodity.

models in which the above-mentioned interdependence is allowed.[5] Therefore, we attempt to examine their critical arguments with respect to Sraffa (1960). We assume general convex economies with non-increasing returns to scale and attempt to specify the conditions under which the invariable measure of value and linearity of income distribution can be obtained.

This paper is organised as follows: In Section 2, we present a brief review of the concept of Ricardo's invariable measure of value and Sraffa's standard commodity. Subsequently, we briefly review the history of the debates on the standard commodity and the linear relation of income distribution that Sraffa derived. In Section 3, we discuss a generalisation of the standard commodity to more general models than Sraffa's (1960). As a result, we obtain the conditions under which the invariable measure of value and linearity of income distribution are maintained even in a rigorous general convex economy. In Section 4, we present our concluding remarks.

2 The Invariable Measure of Value and Debates concerning Sraffa (1960)

In this section, we briefly review the concept of Ricardo's invariable measure of value and Sraffa's standard commodity. We also review the linear relation of income distribution, which is obtained in Sraffa (1960), and the debates concerning the significance of the standard commodity and linearity of income distribution.

2.1 Ricardo's invariable measure of value

A measure of value is indispensable for exchanging commodities efficiently. Ricardo asserted that the conditions necessary to make a measure of value perfect are that it should itself have a value, and that value should itself be invariable (Ricardo, 1951D, p. 361). Concerning the first condition, he clearly argued that the labour content embodied in such a commodity represents the exchange value of the commodity. The second condition, the invariance of the value of such a commodity, perplexed him throughout his life.

Why is it difficult to obtain an invariable measure of value? First, the technique to produce it must remain unchanged. In other words, a commodity eligible to become the invariable measure of value is one 'which now and at all times required

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[5]One of the exceptions is Yagi (2012). Following Pasinetti (1981, 1993), he constructed a model in order to compare two different economic systems (called Period 1 economy and Period 2 economy). In addition, he investigated the invariable measure of value and linearity of income distribution.

precisely the same quantity of labour to produce it.' However, Ricardo realised, 'Of such a commodity we have no knowledge, and consequently are unable to fix on any standard of value' (Ricardo, 1951A, p. 17, n. 3). In fact, Ricardo regarded money (that is, gold and silver) as the invariable measure of value, but it is just 'as near as approximation to a standard measure of value as can be theoretically conceived' (Ricardo, 1951A, p. 45). The justification is based on his recognition that the techniques of production of gold and silver are subject to fewer variations (Ricardo, 1951A, p. 87).

With respect to the second condition, even though the technique to produce gold and silver is unchanged, gold and silver cannot be the invariable measure of value. This is because all industries have different proportions of capital and labour, different proportions of circulating and fixed capital, different degrees of durability of fixed capital, and different time-periods necessary to bring the commodity to market. In this situation, the change in the level of wage rates causes changes in relative prices. Furthermore, as already mentioned, we cannot precisely measure the change in prices of the commodities measured, because the prices of gold and silver themselves are subject to the relative variations. Therefore, the invariable measure of value never existed in reality. According to Ricardo (1951A, p. 45), however, the effect of a change in income distribution on relative prices is smaller than the effect of a change in technique. Therefore, Ricardo thought of the deviation of value from the embodied quantity of labour as sufficiently slight (Ricardo, 1951B, p. 66), and thus, he was reluctantly content to say that money can be regarded as the invariable measure of value at the first approximation.

2.2 Sraffa's standard commodity and income distribution

Sraffa (1960) revived the concern about the invariable measure of value, which in turn led to intensive and comprehensive controversies about capital and income distribution.[6] As already mentioned, Ricardo had defined the conditions that the invariable measure of value should satisfy: the invariance of the measure of value with respect to changes in both income distribution and technique. Ricardo was perplexed by the conditions, because he attempted to solve the two simultaneously. In contrast, Sraffa concetnrated on finding a measure of value that is invariable with respect to a change in income distribution, left aside the chnage in techniques. Furthermore, it is Sraffa's breakthrough idea to find a special, composite commodity, termed the *standard commodity*, which plays the role of the invariable measure of value; whereas Ricardo attempted to find a single commodity that plays the role.

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[6]See, for example, Harcourt (1972) for more detail on these controversies.

Let us briefly review the concept of the standard commodity in a single production system. The price system is defined as follows:

$$\mathbf{p} = (1 + \pi)\,\mathbf{p}\boldsymbol{A} + w\boldsymbol{L}, \tag{1}$$

where $\mathbf{p}$, $\boldsymbol{L}$, and $\boldsymbol{A}$ denote the price vector, the labour coefficient vector, and the input coefficient matrix, respectively. For the sake of simplicity, $\boldsymbol{A}$ is assumed to be an indecomposable and productive matrix. π and w denote the rate of profit and the wage rate, respectively. In order to escape from the impasse that Ricardo faced, Sraffa attempted to find an (imaginary) industry that has a value-ratio of the net product to means of production such that the increase in profit is exactly offset by the decrease in wage when the wage rate is reduced. The value-ratio can be obtained by solving the following system:

$$
\begin{aligned}
(1 + \Pi)\,\boldsymbol{A}\mathbf{q} &= \mathbf{q}, \\
\boldsymbol{L}\mathbf{q} &= 1,
\end{aligned}
$$

where $\mathbf{q}$ is the vector denoting the output level of the industry that has the value-ratio. Since we assume the indecomposability of $\boldsymbol{A}$, the above system of equations has the solution of $\Pi^* > 0$ and $\mathbf{q}^* > 0$ from the Perron-Frobenius theorem.[7] The above system of equations is called the standard system. Π^* is the value-ratio termed the standard ratio, which is related to the Frobenius root λ_A as $\lambda_A = \frac{1}{1+\Pi^*}$. Furthermore, $\mathbf{q}^*$ is the corresponding eigenvector, termed the standard commodity. By substituting the solution into the system, we obtain:

$$
\begin{aligned}
(1 + \Pi^*)\,\boldsymbol{A}\mathbf{q}^* &= \mathbf{q}^*, \tag{2} \\
\boldsymbol{L}\mathbf{q}^* &= 1. \tag{3}
\end{aligned}
$$

From formula (2), we obtain:

$$\frac{\mathbf{p}\,[\boldsymbol{I} - \boldsymbol{A}]\,\mathbf{q}^*}{\mathbf{p}\boldsymbol{A}\mathbf{q}^*} = \Pi^*, \tag{4}$$

where $\boldsymbol{I}$ denotes the identity matrix. Formula (4) means that the ratio of the net product to means of production, measured by the standard commodity, is always

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[7]See Pasinetti (1977, pp. 95–7) in detail.

constant, irrespective of price variations.[8] Therefore, Π^* is a real ratio that is independent of prices. Sraffa defined the standard net product and chose it as the numéraire as follows:

$$\mathbf{p}\left[\boldsymbol{I} - \boldsymbol{A}\right]\mathbf{q}^* = 1. \tag{5}$$

Note that the standard commodity $\mathbf{q}^*$ does not need to be actually produced; it is a "purely auxiliary construction" (Sraffa, 1960, p. 31). Sraffa (1960, p. 26) said that any actual economic system can always be transformed into the standard system. The ratio that keeps the net product to means of production constant irrespective of prices is always, so to speak, 'hidden' within any actual economic system. In Sraffa's model, nothing except income distribution ever changes; the technique in use, output level, and proportion of means of production to labour are all fixed. Therefore, no assumption on returns to scale needs to be made here, as Sraffa (1960, p. v) himself said. Under such assumptions, he exclusively analysed the change in relative prices caused by the change in income distribution. Owing to formulae (4) and (5), there is no need for a variation in the price of $\mathbf{q}^*$ to restore the surplus or deficit in the (imaginary) industry which produces the commodity, when the wage rate is reduced. Therefore, the variation in relative prices caused by a change in income distribution is solely attributed to the variation in prices measured on the basis of the invariance property of the numéraire defined by the standard commodity.

The advantage of adopting the standard commodity as the numéraire is not only that it plays the role of the invariable measure of value, but also that it shows us the useful relation of income distribution. From (1) and (5), we obtain:

$$\pi = \Pi^*\left(1 - w\right). \tag{6}$$

Here, w denotes the wage rate or the wage share in terms of the standard commodity,

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[8]According to Schefold (1986, 1989), the watershed condition and the recurrence condition must hold in order for the standard commodity to serve as the invariable measure of value. The former means that the industry producing the standard commodity must adopt that proportion of means of production to labour in which there is no need to change its price when income distribution changes, because the industry neither earns a surplus nor incurs a deficit. The latter condition means that the proportion 'recurs in all the successive layers of the industry's aggregate means of production without limit' (Sraffa, 1960, p.16). In other words, the conditions imply that not only the industry producing the standard commodity, but also all the industries that produce the means of production necessary for the production of the standard commodity, must adopt the 'watershed' proportion of means of production to labour. The industry producing $\mathbf{q}^*$ obviously satisfies both conditions. As is easily shown, the proportion is $\frac{1}{\Pi^*}$. In fact, only the watershed condition is sufficient for the existence of the standard commodity, insofar as the proof is based on the Perron-Frobenius theorem.

whereas π is the actual rate of profit. The distributional relation is expressed by the straight line in the case where the 'organic composition' of capital is not equal in all industries, if and only if the standard commodity is adopted as the numéraire. The important implication of function (6) is that the rate of profit can be obtained without knowing prices, once we know the wage in terms of the standard commodity. In other words, the standard commodity enables us to deal with income distribution independently of prices. As Pasinetti (2006, p. 154) pointed out, the relevance of function (6) does not lie in its linearity, but in the fact that it is independent of prices.

We can conclude that Sraffa resolved the problem that Ricardo could not, but the resolution was *partial*, because Sraffa did not consider another problem. This is the problem of the measure of value invariable with respect to the change in technique.

2.3   After Sraffa (1960)

There have been many reactions to Sraffa (1960) since its publication and some debates on the results that Sraffa derived. The debates focused not only on the invariable measure of value, but also on the usefulness of the standard commodity and the distributional relation given by function (6). Some arguments appreciate Sraffa's achievements, especially his contribution of constructing the standard commodity as the invariable measure of value (for example, Roncaglia, 1978). Other arguments are critical of Sraffa (1960). First, some economists argued that the standard commodity does not play the role of the Ricardian invariable measure of value. Those arguments pointed out the flaw in Sraffa's analysis. Flaschel (1986) is a typical example. The second critical argument was that the standard commodity and the distributional relation shown by function (6) are so restrictive that they are not too helpful for relevant analyses. Those arguments were mainly raised by neoclassical economists, who were interested in variations in output and proportions of means of production.

Let us examine Flaschel (1986) first. According to him, there is a specific and complete solution to the problem of determining the conditions for the invariable measure of value, but Sraffa's standard commodity does not fulfil those conditions. It seems to us that his definition of invariance is different from those of Ricardo and Sraffa. He defined that given $\mathbf{e} - \boldsymbol{A}\mathbf{e}$ as the numéraire, where $\mathbf{e}$ is a vector, all the elements of which are units, an arbitrary composite commodity $\mathbf{b}$ has the invariance property if and only if $\mathbf{pb} = 1$ holds for any non-negative and non-zero $\mathbf{p}$, with $\mathbf{p}\left[\boldsymbol{I} - \boldsymbol{A}\right]\mathbf{e} \equiv 1$ (Flaschel, 1986, pp. 597–8).[9] Certainly Sraffa (1960, p.11) adopted

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[9]In Flaschel (1986, p. 597), it is explicitly written as 'the problem of invariance cannot be described unless a measure of value has already been assumed. This fact is implicitly taken into

$[\boldsymbol{I} - \boldsymbol{A}]\,\mathbf{e}$ as the numéraire, but the numéraire adopted in the context is irrelevant to the issue of the invariable measure of value, and his arguments on the standard commodity have nothing to do with the numéraire of $[\boldsymbol{I} - \boldsymbol{A}]\,\mathbf{e}$. Flaschel's critique of the standard commodity, therefore, seems pointless.[10]

As for the second argument critical of Sraffa, the typical and early example is Burmeister (1968), in which the results obtained by Sraffa (1960) were rigorously formulated. The conclusions he derived are summarised as follows:

1) It is dubious what economic significance can be attached to the standard commodity.

2) The linearity of the distributional relation does not hold if wages are paid at the beginning of the production period rather than at the end.

3) Without the assumption of constant returns to scale and a fixed coefficients matrix, Sraffa's analysis is meaningless if the quantity produced by an arbitrary industry changes.

After Burmeister (1968), he published a set of papers related to Sraffa's analysis and repeated conclusions similar to those above (Burmeister, 1975, 1977, 1980, 1984). However, his conclusions show that he misunderstood some aspects of Sraffa (1960).

The first conclusion made by him is a serious misunderstanding. Burmeister regarded the standard commodity as the *actual* consumption basket by which the real wage rate w in function (6) is measured.[11] Therefore, he argued that the standard commodity has no economic significance; 'Sraffa's weights used to construct his basket of goods are seen to be determined completely from the technology without regard for consumption preferences' (Burmeister, 1984, p. 509). However, the adoption of the standard commodity as the numéraire does not imply that people must actually consume each commodity in the same proportion as that given by the standard commodity.[12] Moreover, it does not imply that each commodity is actually produced in the same proportion as that given by the standard commodity (see Kurz and Salvadori, 1987, pp. 876–7).

The second conclusion is obviously correct. Using the same notations as before,

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account by Sraffa ([1960], Ch. 3) in his assumption $\mathbf{p}\,(\mathbf{e} - \boldsymbol{A}\mathbf{e}) \equiv 1$. $\cdots$ the search for (conditions for) a "measure of value" relative to an already given measure of value! But what can be expected from the solution of such a problem?'

[10]Concerning Flaschel's (1986) critique, see also Baldone (2006) and Bellino (2004), in which the conditions for the invariable measure of value in Ricardo's and Sraffa's sense are adequately formulated.

[11]Samuelson (2008) also blundered into the same misinterpretation as Burmeister.

[12]See Pasinetti (1977, p. 116, n. 40). Any proportion is feasible in the actual consumption basket, even when the standard commodity is adopted as the numéraire.

the distributional relation in this case is expressed as follows:

$$\pi = \frac{\Pi^*}{1 + \Pi^* w} \left(1 - w \right).^{13}$$

(7)

Although it is a hyperbolic function, the basic property obtained when the standard commodity is chosen as the numéraire is still intact: the distributional relation shown by function (7) is entirely independent of prices.

The third conclusion is controversial. Samuelson (2000) and Samuelson and Etula (2006) also argued that constant returns to scale is an indispensable assumption in order to retain the significance of Sraffa's analysis. Samuelson (2000, p. 123) stated that 'if a Sraffian denies constant returns to scale, the one-hundred-page 1960 classic evaporates into a few paragraphs of vapid chit-chat.' Against these arguments, some proponents of Sraffa argued that the assumption on returns to scale is unnecessary in Sraffa's analysis. The characteristic of the analysis is that it is based on the classical surplus approach, which is clearly expressed in Ricardo (1951A). In the approach, the analysis of the distribution of physical surplus comes first. Therefore, Eatwell (1977) emphasised the difference in the analytical basis between classical and neoclassical economics. In the former, the size and composition of output, technique in use, and real wage are the *data*, on the basis of which the distribution of surplus, price formulation, and quantities of input and labour employed are obtained. In the latter, on the contrary, the preferences of individuals, initial endowment of commodities and/or factors of production, distribution of the initial endowments among individuals, and technology are the *data*, and all variables are determined by the interaction between supply and demand. It is based on the marginal method, and thus the assumption on returns to scale is necessary in neoclassical economics.[14] Eatwell (1977) thus argued that the assumption of constant returns to scale is irrelevant in Sraffa's analysis, because it is based on the classical surplus approach.

However, Burmeister and Samuelson considered what happens to the model when the output level changes. If constant returns to scale are not assumed, the change in output level would cause a change in technique. Therefore, Burmeister (1975) argued that function (6) becomes useless if constant returns to scale are not assumed. As is already mentioned, unless constant returns to scale are assumed, the technique generally changes as the output level changes. Since each coefficient matrix has the specific standard ratio, the standard ratio also changes when the technique in use changes. Therefore, function (6) no longer gives us any useful information on

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[13] See Pasinetti (1977, p. 131).

[14] See also Pasinetti (1977, chap. 1; 1981, chap. 1) concerning the contrast between classical and neoclassical economic thoughts and methods.

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income distribution when a change in the output level causes a change in technique in economies without constant returns to scale.[15] Burmeister (1977, pp. 69–70) thus replied to Eatwell: 'I conclude that constant returns to scale is irrelevant for Sraffa's analysis only if one is content to pose irrelevant questions.'

Although it is true that Burmeister's interpretation of Sraffa included the misunderstanding, it is also true that he raised important questions on income distribution which Sraffa had not addressed. The questions are whether or not the invariable measure of value exists in economies where not only income distribution but also technical change are available; and if it exists, what kind of relationship between the invariable measure of value and income distribution holds. We think it worthwhile to examine them. From the viewpoint of modern economic theories, these are, in fact, natural questions, because nearly all modern economic theories allow for interdependence among changes in income distribution, output level, and techniques. In fact, Sraffa (1925, 1926) himself had considered the relationship between returns to scale and choice of techniques, although his consideration was related to the critique of Marshallian partial equilibrium analysis.

3   The Standard Commodity and Income Distribution

In this section, we focus on the questions raised by Burmeister, Mainwaring, and Samuelson. In other words, we investigate the conditions for obtaining the invariable measure of value and maintaining the linearity of income distribution in general convex economies with non-increasing returns to scale.

3.1   Preliminary

Let $\mathbb{R}_+$ be the set of all non-negative real numbers, and $\mathbb{R}_{++}$ be the set of all positive numbers. Let $\mathbb{R}_+^n$ (resp. $\mathbb{R}_{++}^n$) be the n-fold Cartesian product of $\mathbb{R}_+$ (resp. $\mathbb{R}_{++}$). For any $x, y \in \mathbb{R}_+^n$, we write $x \geqq y$ to mean $[x_i \geqq y_i$ for all $i = 1, \ldots, n]$, $x \geq y$ to mean $[x_i \geqq y_i$ for all $i = 1, \ldots, n$ and $x \neq y]$, and $x > y$ to mean $[x_i > y_i$ for all $i = 1, \ldots, n]$.

Let there be N agents in the economy, and let us use N as the population in the economy with generic element $\nu \in N$. Let there be n commodities which are reproducible. Let $\mathbf{0}$ denote the null vector. Production technology is freely available

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[15]Mainwaring (1979), a non-neoclassical economist, also argued that Sraffa's analysis is not valid without the assumption of constant returns to scale or other restrictive assumptions.

to all agents, who can operate any activity in the production set P, which has elements of the form $\alpha = (-\alpha_l, -\underline{\alpha}, \overline{\alpha})$, where $\alpha_l \in \mathbb{R}_+$ is the *effective* labour input of the process; $\underline{\alpha} \in \mathbb{R}_+^n$ are the inputs of the produced goods used in the process; and $\overline{\alpha} \in \mathbb{R}_+^n$ are the outputs of the n goods. Thus, elements of P are vectors in $\mathbb{R}^{2n+1}$. Let P satisfy the following assumptions.

Assumption 0 (A0). P is closed in $\mathbb{R}^{2n+1}$ and $\mathbf{0} \in P$. Moreover, for any $\alpha, \alpha' \in P$ with $\overline{\alpha} = \overline{\alpha}'$, and for any $t \in [0,1]$, $t\alpha + (1-t)\alpha' \in P$.

Assumption 1 (A1). For all $\alpha \in P$, if $\overline{\alpha} \geq \mathbf{0}$, then $\alpha_l > 0$ and $\underline{\alpha} \geq \mathbf{0}$.

Assumption 2 (A2). For all $c \in \mathbb{R}_+^n$, there is a $\alpha \in P$ such that $\widehat{\alpha} \equiv \overline{\alpha} - \underline{\alpha} \geqq c$.

Assumption 3 (A3). For all $\alpha \in P$, and for all $(-\alpha_l', -\underline{\alpha}', \overline{\alpha}') \in \mathbb{R}_- \times \mathbb{R}_-^n \times \mathbb{R}_+^n$, if $(-\alpha_l', -\underline{\alpha}', \overline{\alpha}') \leqq \alpha$, then $(-\alpha_l', -\underline{\alpha}', \overline{\alpha}') \in P$.

Assumption 4 (A4). There exists $r > 0$ such that for all $\alpha \in P$, and for any $k > 0$, $(-k\alpha_l, -k\underline{\alpha}, k^r\overline{\alpha}) \in P$.

For each production possibility set P, let us denote $\partial P \equiv \{\alpha \in P \mid \nexists \alpha' \in P : \alpha' > \alpha\}$ and $SP \equiv \{\alpha \in P \mid \nexists \alpha' \in P : \alpha' \geq \alpha\}$, where the former and the latter are respectively the boundary and the efficiency frontier of the production set P. Moreover, given $k > 0$, let $P(\alpha_l = k) \equiv \{\alpha \in P \mid \alpha_l = k\}$ and

$$\partial P(\alpha_l = k) \equiv \{\alpha \in P(\alpha_l = k) \mid \nexists \alpha' \in P(\alpha_l = k) : (-\underline{\alpha}', \overline{\alpha}') > (-\underline{\alpha}, \overline{\alpha})\}.$$

The model of production sets with **A0~A4** covers a broad class of production technologies as follows:

Example 1: Given a von Neumann technology $(\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{L})$, where $\boldsymbol{A}$ and $\boldsymbol{B}$ are $n \times m$ non-negative matrices and $\boldsymbol{L}$ is a $1 \times m$ positive vector. Suppose that for each sector $j = 1, \ldots, m$, there exists at least one commodity $i = 1, \ldots, n$ such that $a_{ij} > 0$. we can define a production set $P_{(\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{L})}$ as

$$P_{(\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{L})} \equiv \left\{\alpha \in \mathbb{R}_- \times \mathbb{R}_-^n \times \mathbb{R}_+^n \mid \exists x \in \mathbb{R}_+^m : \alpha \leqq (-\boldsymbol{L}x, -\boldsymbol{A}x, \boldsymbol{B}x)\right\}.$$

Note that for each $\alpha \in SP_{(A, \boldsymbol{B}, \boldsymbol{L})}$, there exists $x \in \mathbb{R}_+^m$ such that $\alpha = (-\boldsymbol{L}x, -\boldsymbol{A}x, \boldsymbol{B}x)$. The set $P_{(\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{L})}$ satisfies all of **A0~A4**. As a special case of the von Neumann technology, we can consider the case that $m = n$ and $\boldsymbol{B} = \boldsymbol{I}$, which implies a Leontief technology $(\boldsymbol{A}, \boldsymbol{I}, \boldsymbol{L})$. Then, we can define $P_{(\boldsymbol{A}, \boldsymbol{L})} \equiv P_{(\boldsymbol{A}, \boldsymbol{I}, \boldsymbol{L})}$ as in the definition of $P_{(\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{L})}$. ∎

Example 2: Let us consider a class of Leontief technology $\left\{\left(\boldsymbol{A}^k, \boldsymbol{L}^k\right)\right\}_{k=1,\ldots,m}$, where for each $k = 1, \ldots, m$, $\boldsymbol{A}^k$ is a $n \times n$ non-negative, productive, and indecomposable

matrix and $\boldsymbol{L}^k$ is a $1 \times n$ positive vector, such that for any $k, k' = 1, \ldots, m$, and for any non-negative $n \times 1$ vectors x^k and $x^{k'}$, $\boldsymbol{A}^k x^k = \boldsymbol{A}^{k'} x^{k'}$ implies $x^k = x^{k'}$ and $\boldsymbol{L}^k x^k = \boldsymbol{L}^{k'} x^{k'}$. Given this, we can define a production set $P_{\left(\boldsymbol{A}^k, \boldsymbol{L}^k\right)_{k=1,\ldots,m}}$ as

$$
\begin{aligned}
P_{\left(\boldsymbol{A}^k, \boldsymbol{L}^k\right)_{k=1,\ldots,m}} \quad &\equiv \quad \left\{ \alpha \in \mathbb{R}_- \times \mathbb{R}_-^n \times \mathbb{R}_+^n \mid \exists S \equiv \left\{ k^1, \ldots, k^S \right\} \subseteq \{1, \ldots, m\}, \right. \\
&\left. \exists \left\{ x^{k^s} \right\}_{k^s \in S} \subseteq \mathbb{R}_+^n : \alpha \leqq \left(-\sum_{k^s \in S} \boldsymbol{L}^{k^s} x^{k^s}, -\sum_{k^s \in S} \boldsymbol{A}^{k^s} x^{k^s}, \sum_{k^s \in S} x^{k^s} \right) \right\}.
\end{aligned}
$$

By the supposition of $\left\{ \left(\boldsymbol{A}^k, \boldsymbol{L}^k \right) \right\}_{k=1,\ldots,m}$, the production set $P_{\left(\boldsymbol{A}^k, \boldsymbol{L}^k\right)_{k=1,\ldots,m}}$ satisfies **A0~A4**. ∎

Suppose that each agent can supply at most one unit of labour per production period. Moreover, let $\omega \in \mathbb{R}_+^n$ be the social endowments of commodities. Then, one economy is represented by a list, $\langle N; P; \omega \rangle$.

3.2 The standard commodity in general convex economies

We are now ready to define the standard commodity.

Definition 1: Given an economy $\langle N; P; \omega \rangle$, a *standard commodity* is a vector $y \in \mathbb{R}_{++}^n$, such that there exists $\alpha \in \partial P \left(\alpha_l = 1 \right)$ with (i) $\widehat{\alpha} = y$; (ii) $\frac{y_i}{\alpha_i} = \frac{y_j}{\alpha_j}$ for any $i, j = 1, \ldots, n$; and (iii) there is no $\alpha' \in \partial P \left(\alpha_l = 1 \right)$ with $\frac{\widehat{\alpha}'_i}{\underline{\alpha}'_i} = \frac{\widehat{\alpha}'_j}{\underline{\alpha}'_j}$ for any $i, j = 1, \ldots, n$ and $\frac{\widehat{\alpha}'_i}{\underline{\alpha}'_i} > \frac{y_i}{\underline{\alpha}_i}$ for any $i = 1, \ldots, n$.

The standard commodity defined here is a generalisation of Sraffa's definition. Firstly, the condition $\alpha \in \partial P \left(\alpha_l = 1 \right)$ with (i) $\widehat{\alpha} = y$ implies that the standard commodity can be produced as a net output via a production activity associated with one unit of labour input, which corresponds to equation (3) in Section 2.2. Secondly, condition (ii) of Definition 1 is a generalisation of the condition represented by equation (2) in Section 2.2. Indeed, it can be interpreted as the uniform rate of surplus that is obtained when the wage rate is zero and the whole of the net product is distributed to profits. Thirdly, condition (iii) of Definition 1 is a generalisation of the maximality condition of the uniform surplus rate represented by Π^* of equation (2). Because of these, we can see that Definition 1 is an extension of the Sraffian definition of the standard commodity characterised by equations (2) and (3) to a more general production technology P with **A0~A4**.

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Given this definition, the general existence of the standard commodity can be shown, as the following theorem shows.

Theorem 1: *Let $r \leqq 1$. Then, under $\boldsymbol{A0}\tilde{}\boldsymbol{A4}$, there exists a standard commodity $y^* \in \mathbb{R}_{++}^n$ associated with $\alpha^* \in \partial P\left(\alpha_l = 1\right)$ and $\widehat{\alpha}^* = y^*$.*

The proof of Theorem 1 is relegated to the Appendix.

The above theorem does not necessarily imply the unique existence of the standard commodity, though its uniqueness is not essential in the following analysis. In contrast, if the production set P is more suitably specified, then the unique existence of the standard commodity can be shown, which will be briefly discussed in the Appendix.

3.3 Linearity of distributive relation and the standard commodity

In this section, we will examine whether or not the standard commodity of the economy $\langle N; P; \omega \rangle$ can function as the invariable measure of value in the economy $\langle N; P; \omega \rangle$. Define the set of price systems by the simplex

$$\triangle^{n+1} \equiv \left\{ (p, w) \in \mathbb{R}_+^{n+1} \mid \sum_{i=1}^n p_i + w = 1 \right\}$$

with generic element (p, w). The simplex is sufficient to provide all the necessary information about price systems, since we are only concerned about the relative prices of all commodities and labour, which are invariant with respect to changes in the numéraire commodity. Furthermore, it is sufficient for the main purpose of our analysis to focus on the change of price systems normalised in the simplex. Indeed, if the price of a commodity bundle is invariable with respect to any change in the price systems normalised in the simplex, it is also invariable with respect to any change in price systems measured by any numéraire. Finally, any commodity bundle would not constitute an n-dimensional simplex as the set of price vectors when it is chosen as the numéraire. This is because the n-dimensional simplex presumes $n+1$ number of components, whereas any commodity bundle is constituted by at most n types of commodities, since it does not contain labour by definition. Hence, without loss of generality, we can analyse any price system independently of the issue of which commodity (bundle) is chosen as the numéraire by focussing on the normalised price vector in the simplex.

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Consider a situation where a price system changes from $(p, w) \in \triangle^{n+1}$ to $(p', w') \in \triangle^{n+1}$. Let π (resp. π') be the maximal profit rate associated with the price system (p, w) (resp. (p', w')). Then, let $\triangle p \equiv p' - p$, $\triangle w \equiv w' - w$, and $\triangle \pi \equiv \pi' - \pi$. In this case, if $\triangle py = 0$ holds for a commodity bundle $y \in \mathbb{R}^n_+ \setminus \{\mathbf{0}\}$, it is not because y is chosen as the numéraire, but because its value is invariable with respect to this change.[16] Then, the following definitions are a generalisation of Baldone (2006):

Definition 2: Given an economy $\langle N; P; \omega \rangle$, let $(p, w) \in \triangle^{n+1}$ and $(p', w') \in \triangle^{n+1}$ be two different equilibrium prices, and π and π' the respectively associated maximal profit rates. Then, a commodity bundle $y \in \mathbb{R}^n_+ \setminus \{\mathbf{0}\}$ serves as the *invariable measure of value with respect to change from (p, w) to (p', w')*, if and only if there exist $x \in \mathbb{R}^n_+ \setminus \{\mathbf{0}\}$ and $k > 0$, such that $(-k, -x, x + y) \in P$ and $\triangle py = 0$ holds whenever this price change involves a *redistribution between profit and wage*, namely, $\triangle \pi px + \triangle wk = 0$.

Definition 3: Given an economy $\langle N; P; \omega \rangle$, a commodity bundle $y \in \mathbb{R}^n_+ \setminus \{\mathbf{0}\}$ serves as the *invariable measure of value*, if and only if for any different equilibrium prices $(p, w) \in \triangle^{n+1}$ and $(p', w') \in \triangle^{n+1}$, it serves as the invariable measure of value with respect to change from (p, w) to (p', w').

That is, a commodity bundle serves as the invariable measure of value, if and only if for any change in the price system involving a redistribution of profit and wage, the price of this commodity bundle is invariable. More precisely speaking, let us consider counterfactually a change in factor prices from (π, w) to (π', w'), while keeping the commodity price vector p constant, such that the increase (*resp.* decrease) in profit is exactly equal to the decrease (*resp.* increase) in wage in the production process of the targeted commodity bundle y. Such a change may be derived from a purely political conflict on the income distribution between capital and labour, or it may involve a change in technique. In any case, however, it may result in a change in commodity prices from p to p'. Then, the commodity bundle y can serve as the invariable measure of value with respect to the change from (p, w) to (p', w') whenever $py = p'y$. Furthermore, if the commodity bundle satisfies such an invariable property for any change in price systems with its corresponding redistribution between wage and profit, it can serve as the invariable measure of value.

It is worth emphasising that in the above definitions, the invariable property must hold regardless of the causality of such a price change. For instance, even if the price change is generated *owing to technical change* so that the selected production

---

[16]Remember that, since price systems are normalised in the simplex, no commodity (bundle) is selected as the numéraire in this analysis.

activity is changed in equilibrium, the value of the commodity bundle is required to be invariable. Note that the possibility of price change due to technical change should not be a concern whenever the production set is given by a simple Leontief technology $P = P_{(\boldsymbol{A,L})}$.

Now, we are ready to discuss the necessary and sufficient condition for the standard commodity to serve as the invariable measure of value.

Theorem 2: *Let $r \leqq 1$. Then, under $\boldsymbol{A0}$~$\boldsymbol{A4}$, for any economy $\langle N; P; \omega \rangle$, let us take any equilibrium prices $(p, w) \in \triangle^{n+1}$ and $(p', w') \in \triangle^{n+1}$. Then, the standard commodity y^* associated with $\alpha^* = (-1, -x^*, x^* + y^*) \in \partial P$ serves as the invariable measure of value with respect to change from (p, w) to (p', w'), if and only if there exist non-negative numbers $\delta, \delta' \in \mathbb{R}_+$ such that $py^* = \pi px^* + w - \delta$, $p'y^* = \pi'p'x^* + w' - \delta'$, and $\delta = \delta'$ hold.*

Proof. By Theorem 1, there exists a standard commodity $y^* \equiv \Pi x^*$ with some $\Pi > 0$ in the economy $\langle N; P; \omega \rangle$. Let $(p, w) \in \triangle^{n+1}$ (resp. $(p', w') \in \triangle^{n+1}$) be an equilibrium price with the associated maximal profit rate π (resp. π'). By definition, $y^* = \Pi x^*$. Since $\alpha_l^* = 1$, $py^* \leqq \pi px^* + w$ and $p'y^* \leqq \pi'p'x^* + w'$ generally hold. Therefore, there are non-negative numbers $\delta, \delta' \in \mathbb{R}_+$ such that $py^* = \pi px^* + w - \delta$ and $p'y^* = \pi'p'x^* + w' - \delta'$ hold. Then,

$$\triangle p \left(x^* + y^*\right) = \left(1 + \pi + \triangle\pi\right) \triangle px^* + \left(\triangle\pi px^* + \triangle w\right) - \left(\delta' - \delta\right).$$

Since $y^* = \Pi x^*$, the above equation can be reduced to

$$\triangle py^* = \frac{1}{\Pi} \left(\pi + \triangle\pi\right) \triangle py^* + \left(\triangle\pi px^* + \triangle w\right) - \left(\delta' - \delta\right).$$

Thus, we have

$$\triangle py^* = \left[1 - \frac{1}{\Pi} \left(\pi + \triangle\pi\right)\right]^{-1} \left(\left(\triangle\pi px^* + \triangle w\right) - \left(\delta' - \delta\right)\right).$$

Suppose that y^* serves as the invariable measure of value with respect to a change from (p, w) to (p', w'). Then, by Definition 2, $\triangle\pi px^* + \triangle w = 0$ implies $\triangle py^* = 0$. Then, by the above second equation, $\delta' - \delta = 0$ must hold.

Conversely, let there be $\delta, \delta' \in \mathbb{R}_+$ such that $py^* = \pi px^* + w - \delta$, $p'y^* = \pi'p'x^* + w' - \delta'$, and $\delta = \delta'$ hold. Then, the above last equation implies that $\triangle py^* = 0$ follows from $\triangle\pi px^* + \triangle w = 0$. Thus, by Definition 2, y^* serves as the invariable measure of value with respect to a change from (p, w) to (p', w'). ∎

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Theorem 2 provides the necessary and sufficient condition for the standard commodity to serve as the invariable measure of value in general convex economies. Remember that δ (resp. δ') represents the loss of profits generated by operating the production process α^* with one unit of labour input at the equilibrium price (p, w) (resp. (p', w')). Given this, the standard commodity can serve as the invariable measure of value with respect to a change from (p, w) to (p', w'), if and only if the loss of profits generated by producing this commodity does not change due to such a change in prices. Then, it obviously follows that the standard commodity can serve as the invariable measure of value, if and only if the loss of profits generated by producing this commodity is invariable with respect to any change in equilibrium prices. Therefore, the theorem suggests that the Ricardian invariable measure of value cannot exist in general convex economies, since the necessary and sufficient condition can hardly be satisfied. It follows that even the standard commodity cannot serve as the invariable measure of value in such economies.

Let $P(p, w) \equiv \left\{ \alpha \in P \mid \alpha = \arg\max_{\alpha'} \frac{p\overline{\alpha}' - p\underline{\alpha}' - w\alpha'_l}{p\underline{\alpha}'} \right\}$. Then, the following corollary gives us a typical situation where the standard commodity serves as the invariable measure of value.

Corollary 1: *Let $r \leqq 1$. Then, under $A0^\sim A4$, for any economy $\langle N; P; \omega \rangle$, let us take any equilibrium prices (p, w) and (p', w'), such that $\alpha^* \in P(p, w) \cap P(p', w')$ holds. Then, the* standard commodity y^* *serves as the* invariable measure of value *with respect to a change from (p, w) to (p', w').*

Proof. Note that $\alpha^* = (-1, -x^*, x^* + y^*) \in P(p, w) \cap P(p', w')$ implies $py^* = \pi p x^* + w - \delta$ and $p'y^* = \pi' p' x^* + w' - \delta'$ hold for $\delta = 0 = \delta'$. Then, by Theorem 2, the desired result immediately follows. ∎

Note that when the production set is represented by a simple Leontief technology, $P = P_{(A,L)}$, then for any equilibrium price vectors (p, w), it follows that $\alpha^* \in P_{(A,L)}(p, w)$. Thus, y^* can be the invariable measure of value with respect to any change in equilibrium prices. However, in this case, the change in equilibrium prices is generated simply by a change in income distribution with no involvement of technical change, because $P_{(A,L)}$ does not allow producers any technical choice.

From now on, let us assume that the standard commodity y^* is selected as the numéraire, though it would not serve as the invariable measure of value in general. Then, by definition, any price vector $p \in \mathbb{R}^n_+$ of commodities is normalised as $py^* = 1$. Define the set of price vectors measured by the standard commodity as $\triangle^{y^*} \equiv \left\{ (p, w) \in \mathbb{R}^{n+1}_+ \mid py^* = 1 \right\}$. Given such a situation, let us analyse whether and

under what condition the linear distributional relationship between profit and wage is preserved in general convex economies.

Given P, let $r \leqq 1$. Then, P represents a non-increasing returns to scale production technology. Thus, we can obtain the following theorem.

Theorem 3: *Given P with $\boldsymbol{A0}\tilde{}\boldsymbol{A4}$, let $r \leqq 1$. Let $(p', w') \in \triangle^{y^*}$ be an equilibrium price vector associated with a production activity $\alpha' \in \partial P$ and a maximal profit rate $\pi' \geqq 0$ such that*

$$
\begin{aligned}
p'\overline{\alpha}' &= (1 + \pi') \, p'\underline{\alpha}' + w'\alpha'_l \text{ and} \\
p'\overline{\alpha} &\leqq (1 + \pi') \, p'\underline{\alpha} + w'\alpha_l \text{ for all } \alpha \in P.
\end{aligned}
$$

Then, the linear function of income distribution, $\pi' = \Pi \, (1 - w')$ is derived by using the standard commodity y^*, if and only if $p'y^* = \pi'p'x^* + w'$ holds.

Proof. By definition of (p', w'), it is generally true that $p'\,(x^* + y^*) \leqq (1 + \pi')\,p'x^* + w'$. If $p'\,(x^* + y^*) < (1 + \pi')\,p'x^* + w'$, then $\pi' = \Pi\,(1 - w')$ does not hold. Indeed, from $x^* + y^* = (1 + \Pi)\,x^*$, it follows that $p'y^* = \Pi p'x^* < \pi'p'x^* + w'$. Since $p'y^* = 1$, then $\pi' > \Pi\,(1 - w')$ holds. Conversely, let $p'\,(x^* + y^*) = (1 + \pi')\,p'x^* + w'$. Then, since $p'y^* = 1$, $p'y^* = 1 = \pi'p'x^* + w'$. Thus, since $\Pi p'x^* = 1$, $\Pi - \frac{w'}{p'x^*} = \pi'$ holds, which is equivalent to $\pi' = \Pi\,(1 - w')$. $\blacksquare$

The crucial point for the above analysis is whether efficient production of the standard commodity is *feasible* at the actual equilibrium prices. Note that this point is irrelevant in single-product systems such as Leontief production economies and as in Sraffa (1960), where the standard commodity is always produced efficiently at any equilibrium price system. In contrast, our model of general convex economies allows for the possibility of joint production as well as of technical choices, under which the standard commodity may be produced only inefficiently at some equilibrium price system. In such a case, Theorem 3 suggests that the linearity of income distribution no longer holds. Combined with Theorem 2 and Corollary 1, it implies that to preserve the value-invariance property and the linear income distribution property of the standard commodity, efficient production of the standard commodity is another indispensable presumption, which is hidden in the model of single-product systems.

Furthermore, $y^* = \Pi x^*$, which is obtained from Theorem 1, corresponds to equation (2). Therefore, as is pointed out in footnote 8, the standard commodity defined here also satisfies both the watershed and recurrence conditions (Schefold, 1986, 1989). Indeed, here, the 'watershed' proportion of means of production to labour is $\frac{1}{\Pi}$, given normalisation of $\alpha_l^* = 1$. This ratio is preserved not only by the industries

producing y^*, but also by all the industries producing the means of production, x^*, which are necessary for the production of y^*. As is shown by Theorem 2, however, the standard commodity defined here cannot always serve as the invariable measure of value, even though both the conditions are satisfied. This is a striking difference between our standard commodity and Sraffa's (1960).

4   Concluding Remarks

The purpose of this paper is to examine the applicability of the standard commodity to more general cases than Sraffa (1960) assumed. In particular, the paper examines the problems which were not addressed by Sraffa (1960), but by Burmeister, Samuelson, and others. That is, the invariable measure of value and the linearity of income distribution are investigated in a model that allows for interdependence among changes in income distribution, output level, and choice of techniques. To this end, we construct general convex economies with non-increasing returns to scale in which the existence of the standard commodity is guaranteed. The implication of Theorems 2 and 3 can be summarised as follows: as Sraffa (1960, p. v) himself suggested, in economies with Leontief production technology (that is, the case of constant returns to scale without joint production nor technical choice), the standard commodity perfectly serves as the invariable measure of value and the linear relation of income distribution is preserved. Otherwise, the standard commodity cannot generally serve as the invariable measure of value and the linearity of income distribution cannot be preserved. This is because, unlike in the case of Leontief economies, efficient production of the standard commodity is not necessarily possible at the actual equilibrium prices.

According to Kurz (2012, p. 1556), the existence of Π^* in formula (4) confirms the correctness of Ricardo's conjecture: 'The great questions of Rent, Wages, and Profits must be explained by the proportions in which the whole produce is divided between landlords, capitalists, and labourers, and which are not essentially connected with the doctrine of value' (Ricardo, 1952A, p. 194). It implies that the rate of profit can be obtained without knowing the structure of prices. However, our results demonstrate that Ricardo's conjecture is not generally valid. The conjecture is valid only under the same assumptions as imposed on Theorem 3, namely, efficient production of the standard commodity is feasible at the actual equilibrium prices. Otherwise, the rate of profit is not independent of the price structure.

5 Appendix

5.1 Proof of Theorem 1

Proof of Theorem 1: Given $P(\alpha_l = 1)$ which is convex by $r \leqq 1$, let $P_{\alpha_l=1}$ be the minimal closed convex cone containing $P(\alpha_l = 1)$. By definition, $P_{\alpha_l=1}$ is a closed convex cone with $P_{\alpha_l=1}(\alpha_l = 1) = P(\alpha_l = 1)$. If $r = 1$, $P_{\alpha_l=1} = P$. Given $P_{\alpha_l=1}$, let $\overline{P}_{\alpha_l=1} \equiv \left\{ \alpha \in P_{\alpha_l=1} \mid \sum_{i=1,\dots,n} \overline{\alpha}_i = 1 \right\}$. Let $F : P_{\alpha_l=1} \to \mathbb{R}_+$ be such that for each $\alpha \in P_{\alpha_l=1}$, $F(\alpha) = \min_{i=1,\dots,n} \frac{\overline{\alpha}_i}{\underline{\alpha}_i}$ where

$$\frac{\overline{\alpha}_i}{\underline{\alpha}_i} \equiv \begin{cases} 0 & \text{if } \overline{\alpha}_i = 0 \\ +\infty & \text{if } \underline{\alpha}_i = 0 \text{ and } \overline{\alpha}_i > 0. \end{cases}$$

This mapping is continuous and well-defined by **A1**. Note that, by **A2** and **A4**, there exists $\alpha' \in \partial P(\alpha_l = 1)$ such that $\widehat{\alpha}' > \mathbf{0}$. Hence, for $\frac{\alpha'}{\sum_{i=1,\dots,n} \overline{\alpha}_i'} \in \partial \overline{P}_{\alpha_l=1}$, $F\left(\frac{\alpha'}{\sum_{i=1,\dots,n} \overline{\alpha}_i'} \right) > 0$. This implies $\sup_{\alpha \in \overline{P}_{\alpha_l=1}} F(\alpha) > 0$. Suppose that $\sup_{\alpha \in \overline{P}_{\alpha_l=1}} F(\alpha) = +\infty$. Then, there exists a sequence $\{\alpha^k\} \subseteq \overline{P}_{\alpha_l=1}$ such that $\alpha^k \to \alpha^*$ with $\lim_{k \to +\infty} F(\alpha^k) = F(\alpha^*) = \sup_{\alpha \in \overline{P}_{\alpha_l=1}} F(\alpha)$. By definition of F, $F(\alpha^*) = +\infty$ implies that $\alpha^* = (-l, \mathbf{0}, \overline{\alpha}^*)$ for some $l \geqq 0$ and some $\overline{\alpha}^* > \mathbf{0}$. Since $\overline{P}_{\alpha_l=1}$ is closed, $\alpha^* \in \overline{P}_{\alpha_l=1}$. By construction, $\overline{P}_{\alpha_l=1}$ satisfies **A1**, which is a contradiction of $\alpha^* \in \overline{P}_{\alpha_l=1}$. Thus, $\sup_{\alpha \in \overline{P}_{\alpha_l=1}} F(\alpha) < +\infty$. Then, $\sup_{\alpha \in \overline{P}_{\alpha_l=1}} F(\alpha) = \max_{\alpha \in \overline{P}_{\alpha_l=1}} F(\alpha)$. Let $\alpha^* \in \arg\max_{\alpha \in \overline{P}_{\alpha_l=1}} F(\alpha)$. Then, by the cone property, $\frac{\alpha^*}{\alpha_l^*} \in P(\alpha_l = 1)$ and $\frac{\alpha^*}{\alpha_l^*} \in \arg\max_{\alpha \in P(\alpha_l=1)} F(\alpha)$. Hence, without loss of generality, let $\alpha^* \in \arg\max_{\alpha \in P(\alpha_l=1)} F(\alpha)$. Then, $\alpha^* \in \partial P(\alpha_l = 1)$. Since there exists $\alpha' \in \partial P(\alpha_l = 1)$ such that $F(\alpha') > 0$, $\max_{\alpha \in P(\alpha_l=1)} F(\alpha) > 0$ holds, which implies that $\overline{\alpha}^* > \mathbf{0}$.

Define $V \equiv \{\overline{\alpha} - F(\alpha^*)\underline{\alpha} \mid (-1, -\underline{\alpha}, \overline{\alpha}) \in P(\alpha_l = 1)\}$. Then, V is a closed convex set with $V \cap \mathbb{R}^n_{++} = \varnothing$. Then, there exists $p^* \in \mathbb{R}^n_+ \setminus \{\mathbf{0}\}$ such that $p^*[\overline{\alpha} - F(\alpha^*)\underline{\alpha}] \leqq 0$ for all $\alpha \in P(\alpha_l = 1)$ and $p^*z > 0$ for all $z \in \mathbb{R}^n_{++}$. This implies that if there exists $i \in \{1, \dots, n\}$ with $\frac{\overline{\alpha}_i^*}{\underline{\alpha}_i^*} > F(\alpha^*)$, then $p_i^* = 0$. By $p^* \in \mathbb{R}^n_+ \setminus \{\mathbf{0}\}$, there exists $i \in \{1, \dots, n\}$ with $\frac{\overline{\alpha}_i^*}{\underline{\alpha}_i^*} = F(\alpha^*)$ and $p_i^* > 0$. Thus, $p^*[\overline{\alpha}^* - F(\alpha^*)\underline{\alpha}^*] = 0$. Hence, p^* is a supporting vector of $\alpha^* \in \partial P(\alpha_l = 1)$. Let $\alpha^{**} \in P(\alpha_l = 1)$ be such that for each $i \in \{1, \dots, n\}$ with $\frac{\overline{\alpha}_i^*}{\underline{\alpha}_i^*} > F(\alpha^*)$, $(\underline{\alpha}_i^{**}, \overline{\alpha}_i^{**}) \in \mathbb{R}^2_{++}$ with $\frac{\overline{\alpha}_i^{**}}{\underline{\alpha}_i^{**}} \equiv F(\alpha^*)$. (Note that such a construction is possible by **A3**.) Furthermore, for each $i \in \{1, \dots, n\}$ with $\frac{\overline{\alpha}_i^*}{\underline{\alpha}_i^*} = F(\alpha^*)$, $(\underline{\alpha}_i^{**}, \overline{\alpha}_i^{**}) \equiv (\underline{\alpha}_i^*, \overline{\alpha}_i^*)$. Then, by construction, $p^*[\overline{\alpha}^{**} - F(\alpha^*)\underline{\alpha}^{**}] = 0$, which implies that $\alpha^{**} \in \partial P(\alpha_l = 1)$. Note that $\underline{\alpha}^{**} > \mathbf{0}$ and $\overline{\alpha}^{**} = F(\alpha^*)\underline{\alpha}^{**}$.

Let $y^* \equiv \overline{\alpha}^{**} - \underline{\alpha}^{**}$. Remember that there exists $\alpha' \in \partial P(\alpha_l = 1)$ such that

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$\widehat{\alpha}' > \mathbf{0}$ and $F(\alpha') > 0$, which implies $F(\alpha^*) \geqq F(\alpha') > 1$. Therefore, $y^* > \mathbf{0}$. Then, there exists a positive number $\Pi > 0$ such that $\Pi x^* = y^*$ for $x^* \equiv \underline{\alpha}^{**} > \mathbf{0}$. By Definition 1, $y^* > \mathbf{0}$ is a standard commodity of the economy $\langle N; P; \omega \rangle$. ∎

5.2 A note on the unique existence of the standard commodity

As discussed in section 3.2, if the general production set P is properly restricted, then the unique existence of the standard commodity can be shown by applying the non-linear Frobenius Theorem à la Fujimoto (1979, 1980). Let us briefly see what kinds of restrictions are required.

Firstly, let us introduce the following additional assumption:

Assumption 5 (A5). For any $\alpha \in \partial P (\alpha_l = 1)$ and any $\underline{\alpha}' \in \mathbb{R}_+^n$ with $\underline{\alpha}' \geq \underline{\alpha}$ and $\underline{\alpha}' \not> \underline{\alpha}$, there exists $(-\alpha_l', \overline{\alpha}') \in \mathbb{R}_- \times \mathbb{R}_+^n$ with $\alpha_l' \geqq 1$ and $(-\alpha_l', -\underline{\alpha}', \overline{\alpha}') \in \partial P$, such that $\overline{\alpha}' \geq \overline{\alpha}$; and there exists at least one commodity $i = 1, \ldots, n$ with $\underline{\alpha}_i' = \underline{\alpha}_i$, such that $\overline{\alpha}_i' > \overline{\alpha}_i$.

A5 is a generalisation of the indecomposability assumption formulated by Fujimoto (1979, 1980) in the context of the non-linear Frobenius Theorem. The *indecomposability* of the economy implies that any commodity is directly and/or indirectly used as an input in producing other commodities. Though **A5** simply focuses on the production points with one unit of labour input, such an interpretation is sensible. Indeed, the indecomposability property stipulated by **A5** is preserved not only in the case of production points in $\partial P (\alpha_l = 1)$, but also in the case of any production points in $\partial P (\alpha_l = k)$ with any $k > 0$. This is guaranteed by the property of homogeneity of degree $r > 0$ of each production set assumed by **A4**.

Secondly, for each commodity $i = 1, \ldots, n$, let $H_i : \mathbb{R}_+^n \to \mathbb{R}_+$ be a continuous and quasi-concave function, such that for each $x \in \mathbb{R}_+^n$, $H_i(x) \in \mathbb{R}_+$ with $H_i(\mathbf{0}) = 0$ and $H_i(x) \geqq H_i(x')$ if $x \geqq x'$. Moreover, for each commodity $i = 1, \ldots, n$, H_i is homogenous of degree r where $0 < r$. Let $H \equiv (H_1, \ldots, H_n)$ be such that for each $x \in \mathbb{R}_+^n$, $H(x) = (H_1(x), \ldots, H_n(x))$. Suppose that for any $x \in \mathbb{R}_+^n$, $x \geq \mathbf{0}$ implies $H(x) \geq \mathbf{0}$, and for any $x, x' \geq \mathbf{0}$ with $x \geq x'$ but $x \not> x'$, it follows that $H(x) \geq H(x')$ and there exists at least one commodity $i = 1, \ldots, n$ with $x_i = x_i'$ such that $H_i(x) > H_i(x')$. Let $L : \mathbb{R}_+^n \to \mathbb{R}_+$, which is a continuous and strongly increasing function, such that for each $x \in \mathbb{R}_+^n$, $L(x) \in \mathbb{R}_+$ with $L(\mathbf{0}) = 0$ and $L(x) > 0$ for any $x \geq \mathbf{0}$. Furthermore, for any $x \geq \mathbf{0}$ and any $k > 0$, $L(kx) = kL(x)$

holds. Then, given a pair of (H, L), we can define a production set $P_{(H,L)}$ as

$$P_{(H,L)} \equiv \left\{ \alpha \in \mathbb{R}_- \times \mathbb{R}_-^n \times \mathbb{R}_+^n \mid \exists x \in \mathbb{R}_+^n : \alpha \leqq (-L(x), -x, H(x)) \right\}.$$

This $P_{(H,L)}$ satisfies the above $\textbf{(A0)}^\sim\textbf{(A5)}$ since H is homogenous of degree r. Thus, if $r < 1$, then $P_{(H,L)}$ exhibits a production set of decreasing returns to scale.

When the production set P is specified by $P = P_{(H,L)}$, it is possible to show that the standard commodity uniquely exists in the economy $\langle N; P_{(H,L).}; \omega \rangle$ by means of the non-linear Frobenius Theorem à la Fujimoto (1979, 1980).

Note that it is possible to show that the production set $P_{\left(\textbf{A}^k, \textbf{L}^k\right)_{k=1,\dots,m}}$ discussed in **Example 2** of Section 3.2 satisfies $\textbf{(A0)}^\sim\textbf{(A5)}$ and can be represented by a profile (H, L) as $P_{(H,L)} = P_{\left(\textbf{A}^k, \textbf{L}^k\right)_{k=1,\dots,m}}$.

It is also possible to find a von Neumann production set $P_{(\textbf{A},\textbf{B},\textbf{L})}$, which is represented by a profile (H, L) with $\textbf{(A0)}^\sim\textbf{(A5)}$.

Example A1: Let us take a von Neumann technology $(\textbf{A}, \textbf{B}, \textbf{L})$ with $m = n$, whose associated $P_{(\textbf{A},\textbf{B},\textbf{L})}$ satisfies $\textbf{(A0)}^\sim\textbf{(A5)}$. Then, for the linear mapping $\textbf{A}$, the *inverse mapping* $A^{-1} : \mathbb{R}_+^n \twoheadrightarrow \mathbb{R}_+^m$ is given by $A^{-1}(\underline{\alpha}) \equiv \left\{ x \in \mathbb{R}_+^m \mid \underline{\alpha} = \textbf{A}x \right\}$ for each $\underline{\alpha} \in \mathbb{R}_+^n$. Then, since $\textbf{A}$ is single-valued, A^{-1} is a linear mapping by Berge (1963, p. 135, Theorem 7.2). Moreover, assume that A^{-1} is single-valued, which implies that $n = m$ and A^{-1} is the inverse matrix of A. Then, define $H \equiv \textbf{B} \circ A^{-1}$ and $L \equiv \textbf{L} \circ A^{-1}$. Thus, for each $\alpha \in SP_{(\textbf{A},\textbf{B},\textbf{L})}$, $H(\underline{\alpha}) = \overline{\alpha}$ and $L(\underline{\alpha}) = \alpha_l$ hold. This implies that $P_{(H,L)} = P_{(\textbf{A},\textbf{B},\textbf{L})}$. Then, if $\textbf{B} \neq \textbf{I}$, $P_{(H,L)}$ represents a production set where joint production is possible.

Moreover, let the matrix $(\textbf{B} - \textbf{A})$ not have the inverse matrix. In this case, for any $c \geq \textbf{0}$, $Rank(\textbf{B} - \textbf{A}, c) < n$ holds, so that there are infinite number of solutions $x \geq \textbf{0}$ satisfying $(\textbf{B} - \textbf{A})x = c$. Then, for a given net output vector c, the set of alternative (efficient) techniques to produce c as a net output is given by

$$\partial \underline{P}_{(\textbf{A},\textbf{B},\textbf{L})}(c) = \left\{ (-\textbf{L}x, -\textbf{A}x) \in \mathbb{R}_- \times \mathbb{R}_-^n \mid \exists x \in \mathbb{R}_+^n : (\textbf{B} - \textbf{A})x = c \right\}.$$

This is equivalent to the following set:

$$\partial \underline{P}_{(H,L)}(c) = \left\{ (-L(\underline{\alpha}), -\underline{\alpha}) \in \mathbb{R}_- \times \mathbb{R}_-^n \mid \exists \underline{\alpha} \in \mathbb{R}_+^n : H(\underline{\alpha}) - \underline{\alpha} = c \right\},$$

by $P_{(H,L)} = P_{(\textbf{A},\textbf{B},\textbf{L})}$. Thus, the system (H, L) can represent economies with joint production as well as those with alternative production techniques.

Instead, if $\textbf{A}$ and $\textbf{B}$ are $n \times m$ matrices with $n < m$, then we cannot define H as a single-valued mapping, though $n < m$ is a natural case in that the number

of alternative production processes is sufficiently large. Indeed, if $n < m$, then $Rank\,(\boldsymbol{A}) < m$. Thus, $A^{-1} : \mathbb{R}^n_+ \twoheadrightarrow \mathbb{R}^m_+$, the inverse of the linear mapping $\boldsymbol{A}$, is not single-valued. Thus, if for some $x, x' \in A^{-1}(\underline{\alpha})$, $\boldsymbol{B}x \neq \boldsymbol{B}x'$ holds, then we cannot define H as a single-valued mapping. Further, if for some $x, x' \in A^{-1}(\underline{\alpha})$, $\boldsymbol{L}x \neq \boldsymbol{L}x'$ holds, then we cannot define L as a single-valued mapping.

For instance, let

$$\boldsymbol{A} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}, \boldsymbol{B} = \begin{bmatrix} 0 & 2 & 1 \\ 2 & 0 & 2 \end{bmatrix}, \boldsymbol{L} = (1,1,1), \text{ and consider } \underline{\alpha} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

Then, for $x = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$ and $x' = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$, $\boldsymbol{A}x = \underline{\alpha} = \boldsymbol{A}x'$, but $\alpha_l = \boldsymbol{L}x = 3 > 2 = \boldsymbol{L}x' = \alpha'_l$, $\overline{\alpha} = \boldsymbol{B}x = \begin{pmatrix} 2 \\ 4 \end{pmatrix} \geq \begin{pmatrix} 1 \\ 4 \end{pmatrix} = \boldsymbol{B}x' = \overline{\alpha}'$, and $\widehat{\alpha} = (\boldsymbol{B} - \boldsymbol{A})x = \begin{pmatrix} 0 \\ 3 \end{pmatrix} \geq \begin{pmatrix} -1 \\ 3 \end{pmatrix} = (\boldsymbol{B} - \boldsymbol{A})x' = \widehat{\alpha}'$. This numerical example has the property that $\boldsymbol{B}x \neq \boldsymbol{B}x'$ and $\boldsymbol{L}x \neq \boldsymbol{L}x'$ hold for some $x, x' \in A^{-1}(\underline{\alpha})$. $\blacksquare$

6   References

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