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Axiomatic characterization of the aggregate consumer surplus measures as social welfare indices

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Abstract: This paper proposes two types of equity-regarding social welfare indices which satisfy the Pigu-Dalton transfer principle in the neoclassical market model: one is the geometrically aggregated compensating ratios and the other is the geometrically aggregated equivalent ratios. The two proposed indices, as well as the well-known two types of arithmetically aggregated Hicksian variations, are used to define Arrovian social ordering functions which evaluate the price vectors and income distributions in the market model. It is shown that each of the four Arrovian social ordering functions can be characterized by the axioms in Arrovian form: Pareto, symmetry and an independence axiom weaker than Arrow's original independence axiom. It follows from the characterizations that the two proposed indices have solid normative foundations as social welfare indices even if the individual heterogeneity both in tastes and in incomes is admitted, and that the equity-regarding property of the indices comes from their respective independence axioms (variants of Fisher's commensurability axiom).

Key words: Equity, Pigu-Dalton transfer principle, Aggregate consumer surplus measures, Arrovian social ordering function, Pareto axiom, Symmetry axiom, Independence axiom.

JEL classification: D63, D61, D11

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1. Introduction

The arithmetically aggregated compensating variations $\sum_i CV^i$ and arithmetically aggregated equivalent variations $\sum_i EV^i$ are adopted in a wide class of policy analyses as normative measures for evaluating the changes of social welfare according to the changes in the price vector and/or the money (wealth) distribution on the consumers in the neoclassical market model. Although the values of the two aggregated measures can in principle be computed only from the market data (individual expenditure functions), the two measures are not *equity-regarding* in the sense that they do not generally satisfy the (fixed-price) Pigu-Dalton principle. Namely, they are not always increased by sufficiently small progressive monetary transfers from consumers with higher income levels to those with lower income levels, as long as the total income is preserved and the price vector is fixed. This disadvantage of the two aggregated measures is significant, since the equity-regarding property of a normative measure has crucial impacts on the outcomes (policy recommendations) of the policy analyses using the measure. For example, the progressiveness of the optimal income tax schedule depends on the equity-regarding property of a normative measure as studied in the literature of the optimal tax theory, including Fleurbaey and Maniquet (2006).

This paper proposes two types of equity-regarding social welfare indices which satisfy the Pigu-Dalton transfer principle[1]: one is the geometrically aggregated compensating ratios $\prod_i CR^i$ and the other is the geometrically aggregated equivalent ratios $\prod_i ER^i$.[2] Moreover, based on the Arrovian axiomatic approach, this paper shows that the two proposed indices have solid normative foundations within the market model, even if the individual heterogeneity both in tastes and in incomes is admitted, which implies that the scope of applicability of the proposed indices is significantly wider than that of the normative evaluation based on the representative consumer,

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[1] Moreover, the product form of the index implies that the "right to live" is justified by the index, i.e., any given money distribution is socially better than the money distribution in which a consumer's money assignment is sufficiently small.

[2] The geometric aggregation like this is introduced by Diewert (1984, Section 5) to construct the democratic price index. The compensating ratio (CR) and equivalent ratio (ER) are introduced by Allen (1949) and Deaton (1979, Section 2) in different terms. The closely related indices are introduced by Hicks (1942, Section 7). Blackorby and Donaldson (1987) study the cost-benefit rule based on the Malmquist-type ratio, which is a variant of the consumer surplus ratios above.

where only the heterogeneity in incomes is admitted.[3]

In Arrow's original setting, a social welfare function is defined independent of the economic systems, and a social welfare function can evaluate possible economic systems as social alternatives in a social contract situation; however, we assume here that the competitive market system (including the private ownership) has been established in the society, and we define a social welfare function in the market by a function which assigns a social ordering on the set of alternatives (pairs of a price vector and an income distribution) to each possible profile of the market specified by not only individual preference orderings but also the consumers' initial holdings of money and initial price vector. Namely, a profile describes a basic market data including an initial competitive equilibrium before the underlying government selects a policy, and an alternative describes an ex-post competitive equilibrium after the government selects a specific policy. By this difference of the definitions, we hereafter use the phrase "social ordering function" instead of "social welfare function", and we attempt to derive the axioms which characterize the social ordering functions determined by the proposed indices. In particular, we want to find the fundamental axioms which imply the Pigu-Dalton transfer principle, without assuming the transfer principle as an axiom. Specifically, not only the two indices, $\prod_i CR^i$ and $\prod_i ER^i$, but also the well-known two arithmetically aggregated Hicksian variations, $\sum_i CV^i$ and $\sum_i EV^i$, are used to define the social ordering functions, which we call CR, ER, CV and EV social ordering functions,[4] and after deriving the axiomatic characterization results for the four social ordering functions we compare the four sets of axioms to find the key axioms which lead to the transfer principle.

It is well-known that the Arrovain impossibility theorem holds even in the market model, i.e., there is no social ordering function which satisfies the three Arrovian axioms: Pareto, symmetry

[3] As shown in Mas-Colell, *et.al.* (1995, Section 4.D, Examples 4.D.2 and 4.D.1), the assumption that all consumers have the same (identical) homothetic preference orderings is needed to ensure the validity of the normative evaluation based on the representative consumer. See also Slesnick (1998, Section 3.1, Pages 2138, 2140 and 2141) and the references.

[4] The geometrically aggregated equivalent variations and arithmetically aggregated equivalent ratios are introduced as counter examples to prove the independence of the axioms in the main theorems (Theorems 3 and 4) of this paper.

and independence axioms on the set of all possible profiles of the market.[5] We attempt to derive a positive result by modifying some of the three axioms and restricting the set of profiles on which the social ordering function is defined.

First, we modify the independence axiom in the Arrovian impossibility theorem, which we call the Arrovian independence axiom. Generally, an independence axiom requires that, if the structures of two profiles coincide for two pairs of alternatives, the social ordering function ought to lead to an identical social ordering between these pairs. For the Arrovian independence axiom, the identical structure of profiles is specified by individual preference orderings only, whereas for each of the four types of (modified) independence axioms introduced here, the identical structure of profiles is specified by numerical values of the corresponding consumer surplus measures: CV^i, EV^i, CR^i and ER^i. Since the (fully cardinal) information in terms of consumer surplus is finer than the (ordinal) information in terms of preference orderings, our independence axioms are weaker than the Arrovian independence axiom, as long as the individual consumer surplus measures are true indices of the individual welfares.[6]

Second, we restrict the set of all profiles, which is the domain of the social ordering functions, by imposing an additional condition on the individual preference orderings. For the CV and EV social ordering functions, the domain is restricted to the quasi-linear domain, where the CV and EV social ordering functions coincide. For the CR and ER social ordering functions, the domain is restricted to the homothetic domain, where the CR and ER social ordering functions coincide.

Under the two modifications, it is shown that each of the four social ordering functions is characterized by the corresponding (modified) independence axiom under the two common axioms, Pareto and symmetry axioms, on the corresponding (restricted) domain, without assuming the

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[5] For the Arrovian impossibility theorem in the market model, see Le Breton and Weymark (2011).

[6] As shown in this paper (Lemma 5), the fully cardinal information can be completely derived by additional (ordinal) information by the preference orderings on all triples including the reference alternative as well as the pair of alternatives, since the values of individual consumer surplus can be computed only by individual preference orderings on the triples. The independence axiom of the Nash social welfare function is introduced by almost the same motivation by Kaneko and Nakamura (1979), except money is replaced with expected utility.

numerical representability of social orderings and interpersonal utility comparisons.[7] Consequently, the CR and ER social ordering functions comes from their respective independence axioms. Moreover, it is also shown that the Arrovian impossibility theorem holds if the domain is enlarged in each of the characterizations,[8] which clarify the scope of the applicability of the proposed indices, $\prod_i CR^i$ and $\prod_i ER^i$, in the economic environment. Since we do not assume any interpersonal condition for consumers' preferences, the scope of applicability of the proposed indices is significantly wider than that of the standard methods evaluating the market outcomes.

In the next section, we introduce a simple economic environment with a finite number of consumers where a social alternative is defined as a distribution of money on the consumers, and we introduce two social ordering functions: the *arithmetic mean* social ordering function which ranks money distributions by the arithmetic mean of the money distribution and the *geometric mean* social ordering function which ranks money distributions by the geometric mean of the money distribution. It is shown that the two social ordering functions are characterized by the three axioms in Arrovian form: Pareto, symmetry and independence axioms (Theorem 1). Section 3 defines the social ordering function in the neo-classical economic environment with m consumption goods, and introduces the four social ordering functions determined by the aggregated consumer surplus measures. Section 4 introduces the axioms in Arrovian form to characterize the four social ordering functions. Although some independence axioms are suitably weakened comparing with the Arrovian independence axiom, the negative result (Theorem 2) is derived when the domain is the full domain. Finally, the domains are restricted to derive positive characterization results, making use of Theorem 1. In Section 5, the CV and EV social ordering functions are axiomatically characterized on the quasi-linear domain (Theorem 3). In Section 6, the CR and ER social ordering functions are axiomatically characterized on the homothetic domain (Theorem 4).

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[7] Jorgenson and Slesnick (1984) and Slivinski (1987) consider the axiomatic foundations of aggregate consumer surplus measures, assuming the specific form of interpersonal utility comparison (cardinal full comparability), or restricting the possible social welfare functions within the class of social welfare functions which can be numerically represented by the (additively separable) Bergson-Samuelson social welfare functions. A comprehensive derivations of these normative measures are given in Ebert and Welsch (2004) and Fleurbaey and Hammond (2004, Section 6).

[8] This implies that the restricted domains are the maximal domains for the characterizations to hold. The concept of maximal domain like this is introduced by Ching and Serizawa (1998) in a different setting.

2. The arithmetic and geometric mean social ordering functions on the set of money distributions

This section introduces social ordering functions in a simple environment with a finite number of consumers and a consumption commodity called money, and this section axiomatically characterizes the two social ordering functions.

There are n ($n \geq 2$) number of consumers, and the set of consumers is denoted by $N = \{1, 2, \cdots, n\}$. There is just one type of consumption commodity called money (numéraire or composite commodity), and i's consumption set is the set of all positive amounts of money denoted by $X_i \equiv \mathbb{R}^1_{++}$. A positive money distribution $\mathbf{x} = (x_1, x_2, \cdots, x_n)$ on N is called an alternative, and the set of all alternatives is denoted by $X \equiv \mathbb{R}^n_{++}$. Each consumer $i \in N$ initially owns an amount of money $x^0_i \in X_i$ and i has a preference ordering on the consumption set X_i. The initial distribution of money is denoted by $\mathbf{x}^0 = (x^0_1, x^0_2, \cdots, x^0_n) \in X$. We assume that i's preference ordering is a complete and transitive binary relation on X_i satisfying the monotonicity. Since there is just one type of preference ordering on X_i for each $i \in N$, a *profile* of society is described by an initial distribution of money $\mathbf{x}^0 \in X$ for simplicity, which implies that the set of profiles coincides with the set of alternatives X. A *social ordering* $\succsim$ is a complete and transitive binary relation on X. The symmetric and asymmetric parts of $\succsim$ are denoted by $\sim$ and $\succ$, respectively. Let $\mathcal{R}$ be the set of all social orderings on X.

A *social ordering function* W is a function $W : X \to \mathcal{R}$ assigning a social ordering $\succsim \in \mathcal{R}$ to each profile (initial money distribution) $\mathbf{x}^0 \in X$, which is a prototype for the social ordering function in the market model introduced in the next section. The symmetric and asymmetric parts of $W(\mathbf{x}^0)$ are denoted by $W_I(\mathbf{x}^0)$ and $W_S(\mathbf{x}^0)$, respectively. Specifically, the *arithmetic mean* social ordering function W^A is defined by

$$\mathbf{x}W^A(\mathbf{x}^0)\mathbf{y} \Leftrightarrow (\Sigma_{i \in N} x_i)/n \geq (\Sigma_{i \in N} y_i)/n \text{ for any } \mathbf{x}^0, \mathbf{x}, \mathbf{y} \in X. \quad (1)$$

The arithmetic mean social ordering $W^A(\mathbf{x}^0)$ simply ranks money distributions by the arithmetic mean of them, independent of the initial money distributions $\mathbf{x}^0$. Since n is a fixed constant, one can alternatively define W^A by

$$\mathbf{x}W^A(\mathbf{x}^0)\mathbf{y} \Leftrightarrow \Sigma_{i \in N} x_i \geq \Sigma_{i \in N} y_i \text{ for any } \mathbf{x}^0, \mathbf{x}, \mathbf{y} \in X. \quad (2)$$

Consequently, the social indifference curves are linear parallel lines in the space of money distributions, and then $W^A(\mathbf{x}^0)$ does not satisfy the Pigu-Dalton transfer principle for all $\mathbf{x}^0$. The *geometric mean* social ordering function W^G on X is defined by

$$\mathbf{x}W^G(\mathbf{x}^0)\mathbf{y} \Leftrightarrow (\Pi_{i\in N} x_i)^{1/n} \geq (\Pi_{i\in N} y_i)^{1/n} \text{ for any } \mathbf{x}^0, \mathbf{x}, \mathbf{y} \in X. \quad (3)$$

The geometric mean social ordering $W^G(\mathbf{x}^0)$ simply ranks money distributions by the geometric mean of them, independent of the initial money distributions $\mathbf{x}^0$. Since n is a fixed constant, one can alternatively define W^G by

$$\mathbf{x}W^G(\mathbf{x}^0)\mathbf{y} \Leftrightarrow \Pi_{i\in N} x_i \geq \Pi_{i\in N} y_i,$$

$$\text{or } \mathbf{x}W^G(\mathbf{x}^0)\mathbf{y} \Leftrightarrow \Sigma_{i\in N} \log x_i \geq \Sigma_{i\in N} \log y_i \text{ for any } \mathbf{x}^0, \mathbf{x}, \mathbf{y} \in X. \quad (4)$$

Hence W^G can be regarded as one of the well-known Atkinson's (1970) social welfare functions parameterized by the degree of inequality aversion. For the next lemma, we need a definition: a money distribution $\mathbf{y}$ is obtained from a money distribution $\mathbf{x}$ by a *progressive transfer* if there exists a pair $i, j \in N$ and $\delta > 0$ such that $x_i - \delta = y_i > y_j = x_j + \delta$ and $x_k = y_k$ for all $k \in N/\{i, j\} \equiv \{l \in N : l \notin \{i, j\}\}$. The following lemma is well-known:

Lemma 1: For any $\mathbf{x}^0, \mathbf{x}, \mathbf{y} \in X$ such that $\mathbf{y}$ is obtained from $\mathbf{x}$ by a progressive transfer, it holds that $\mathbf{y}W^A_I(\mathbf{x}^0)\mathbf{x}$ and $\mathbf{y}W^G_S(\mathbf{x}^0)\mathbf{x}$.

Lemma 1 means that $W^G(\mathbf{x}^0)$ satisfies the *Pigu-Dalton transfer principle* for all $\mathbf{x}^0 \in X$, although $W^A(\mathbf{x}^0)$ does not satisfy the principle for all $\mathbf{x}^0 \in X$.[9]

To characterize the two social ordering functions, let us consider the following axioms:

Pareto: For any $\mathbf{x}^0, \mathbf{x}, \mathbf{y} \in X$, if $x_i > y_i$ for all $i \in N$, then $\mathbf{x}W_S(\mathbf{x}^0)\,\mathbf{y}$.

Symmetry: $\mathbf{x}W_I(\mathbf{x}^0)\,\theta\circ\mathbf{x}$ for any $\mathbf{x}^0, \mathbf{x} \in X$ and any permutation θ of N, where $\theta\circ\mathbf{x} = (x_{\theta(1)}, x_{\theta(2)}, \ldots, x_{\theta(n)})$.[10]

The Pareto axiom is standard. The symmetry axiom requires that the two alternatives are socially indifferent if one is derived by a permutation of the individual money allocations of the other one

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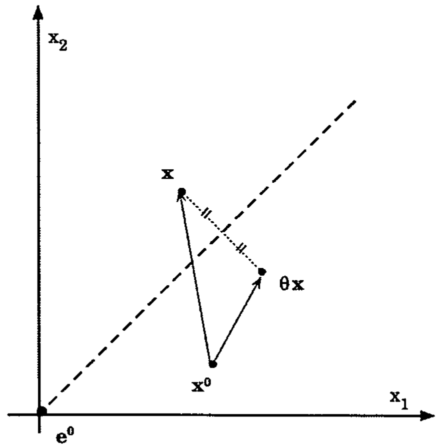
9 See Dutta (2002, Section 3.1) for the Pigu-Dalton transfer principle.

10 A function $f : N \to N$ is called a permutation on N if and only if f is a one-to-one and onto function. The symmetry axiom can not be replaced with a weaker symmetry axiom to characterize the two social ordering functions. See Remark 1 at the end of this section.

independent of the initial money distribution. See Figure 1.

Figure 1: $\mathbf{x} W_I(\mathbf{x}^0) \theta \circ \mathbf{x}$

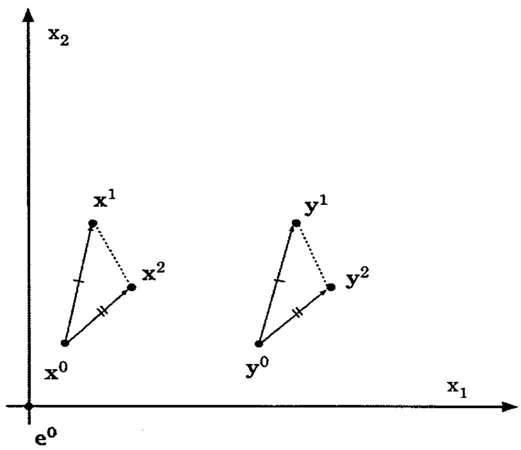


The above two axioms impose the conditions on a social ordering for a given profile. The next axiom is the independence axiom, which specifies the inter-profile consistency of a social ordering function. Practically, an independence axiom requires that, if the (local) structures of two profiles coincide for two pairs of money distributions, the social ordering function ought to lead to identical social ordering between these pairs. Depending on the specification of the identical structure of profiles, the definition of independence axiom has some variations. The first one is the following independence axiom:

A-independence: If $\mathbf{x}^1 - \mathbf{x}^0 = \mathbf{y}^1 - \mathbf{y}^0$ and $\mathbf{x}^2 - \mathbf{x}^0 = \mathbf{y}^2 - \mathbf{y}^0$, then $\mathbf{x}^1 W(\mathbf{x}^0)\, \mathbf{x}^2 \Leftrightarrow \mathbf{y}^1 W(\mathbf{y}^0)\, \mathbf{y}^2$.

The A-independence axiom requires that if the *differences* from the corresponding initial money distributions coincides for two pairs of money distributions, the social ordering function ought to lead to an identical social ordering between these pairs. Consequently, the initial money distribution is a reference point to identify the same (preference) structure for different pairs of alternatives. See Figure 2.

Figure 2
$\mathbf{x}^1 W(\mathbf{x}^0)\, \mathbf{x}^2 \Leftrightarrow \mathbf{y}^1 W(\mathbf{y}^0)\, \mathbf{y}^2$.

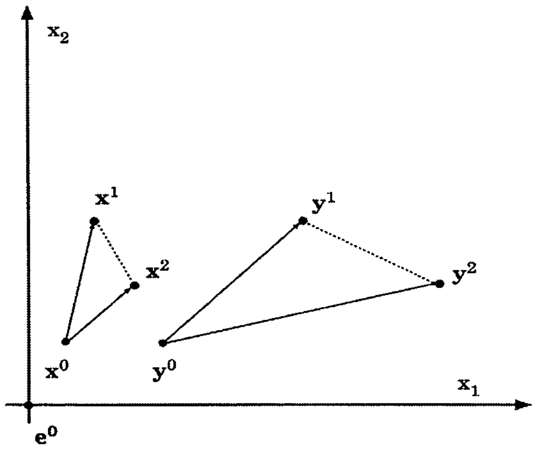


In order to state the next independence axiom, we need a notation: $\mathbf{x}/\mathbf{y} = (x_1/y_1, x_2/y_2, \cdots, x_n/y_n) \in X$ for all $\mathbf{x}, \mathbf{y} \in X$. The next independence axiom is the R-independence axiom:

R-Independence: If $\mathbf{x}^1/\mathbf{x}^0 = \mathbf{y}^1/\mathbf{y}^0$ and $\mathbf{x}^2/\mathbf{x}^0 = \mathbf{y}^2/\mathbf{y}^0$, then $\mathbf{x}^1 W(\mathbf{x}^0)\mathbf{x}^2 \Leftrightarrow \mathbf{y}^1 W(\mathbf{y}^0)\mathbf{y}^2$.

The R-independence axiom requires that if the *ratios* of the money distributions over the corresponding initial money distributions coincides for two pairs of money distributions, the social ordering function ought to lead to an identical social ordering between these pairs.[11] See Figure 3.

Figure 3
$\mathbf{x}^1 W(\mathbf{x}^0)\, \mathbf{x}^2 \Leftrightarrow \mathbf{y}^1 W(\mathbf{y}^0)\, \mathbf{y}^2$.



As a main result of this section, we have the following theorem:

Theorem 1: **(A)** A social ordering function W satisfies the Pareto, symmetry and A-independence axioms if and only if W coincides with the arithmetic mean social ordering function W^A, i.e.,

$$\mathbf{x}W(\mathbf{x}^0)\mathbf{y} \Leftrightarrow (\Sigma_{i\in N} x_i)/n \geq (\Sigma_{i\in N} y_i)/n \text{ for any } \mathbf{x}^0, \mathbf{x}, \mathbf{y} \in X.$$

(B) A social ordering function W satisfies the Pareto, symmetry and R-independence axioms if and only if W coincides with the geometric mean social ordering function W^G, i.e.,

$$\mathbf{x}W(\mathbf{x}^0)\mathbf{y} \Leftrightarrow (\Pi_{i\in N} x_i)^{1/n} \geq (\Pi_{i\in N} y_i)^{1/n} \text{ for any } \mathbf{x}^0, \mathbf{x}, \mathbf{y} \in X.$$[12]

Theorem 1(A) states that the arithmetic mean social ordering function W^A defined by (1) is characterized by the three axioms: Pareto, symmetry and A-independence axioms, and Theorem 1(B) states that the geometric mean social ordering function W^G defined by (3) is characterized by

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[11] In the price index theory, Irving Fisher introduced a closely related axiom called the commensurability test, which requires that the index number should be invariant against the changing the units of measurement of prices or quantities. For the Fisher's commensurability test, see Diewert (2008).

the three axioms: Pareto, symmetry and R-independence axioms. Hence the difference of the two social ordering functions is simply explained by the difference of the independence axioms.

The three axioms in Theorem 1(A) are mutually independent. Practically, constructing some counter examples, we can prove that the Pareto and symmetry axioms each is independent of the other axioms. For the independence of the Pareto axiom, define a social ordering function W^1 by $\mathbf{x}W^1(\mathbf{x}^0)\mathbf{y} \Leftrightarrow \Sigma_{i\in N}x_i \leq \Sigma_{i\in N}y_i$, and for the independence of the symmetry axiom, define a social ordering function W^2 by $\mathbf{x}W^2(\mathbf{x}^0)\,\mathbf{y} \Leftrightarrow 2x_1 + x_2 + \cdots + x_n \geq 2y_1 + y_2 + \cdots + y_n$. It holds by Theorem 1(B) that the social ordering function W^G satisfies the Pareto and symmetry axioms. However W^G does not satisfy the A-independence axiom.[12] Hence the A-independence axiom is independent of the other axioms. Similarly we can prove that the three axioms in Theorem 1(B) are mutually independent.[13]

The Arrovian independence axiom in this setting can be stated as follows:

Arrovian independence: For any $\mathbf{x}^0, \mathbf{x}, \mathbf{y}, \mathbf{y}^0, \mathbf{x}^*$ and any $\mathbf{y}^* \in X$, if $x_i \geq y_i \Leftrightarrow x_i^* \geq y_i^*$ for all $i \in N$, then $\mathbf{x}W(\mathbf{x}^0)\mathbf{y} \Leftrightarrow \mathbf{x}^*W(\mathbf{y}^0)\mathbf{y}^*$.

The A-independence axiom is weaker than the Arrovian independence axiom, since the if-part of the A-independence axiom is strengthened by introducing a stronger condition to identify the same structures of the two profiles on different pairs of alternatives. If there is a social ordering function W satisfying the Pareto, symmetry and Arrovian independence axioms on X, it holds by Theorem 1(A) that W coincides with W^A. Since W^A does not satisfies the Arrovian independence axiom, we have that there is no social ordering function satisfying the Pareto, symmetry and Arrovian independence axioms on X. This means that the Arrovian impossibility theorem holds in this setting.[14]

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[12] Setting $\mathbf{x}^0 = (2, 2, 1, \cdots, 1)$, $\mathbf{y}^0 = \mathbf{x} = (3, 2, 1, \cdots, 1)$, $\mathbf{x}^* = (2, 3, 1, \cdots, 1)$, $\mathbf{y}^* = (4, 2, 1, \cdots, 1)$ and $\mathbf{y} = (3, 3, 1, \cdots, 1)$, we have that $\mathbf{x}W_I^G(\mathbf{x}^0)\,\mathbf{x}^*$ and $\mathbf{y}W_S^G(\mathbf{y}^0)\mathbf{y}^*$, but $\mathbf{x} - \mathbf{x}^0 = \mathbf{y}^* - \mathbf{y}^0 = (1, 0, 0, \cdots, 0)$, and $\mathbf{x}^* - \mathbf{x}^0 = \mathbf{y} - \mathbf{y}^0 = (0, 1, 0, \cdots, 0)$.

[13] For the independence of the Pareto axiom, define a social ordering function W^3 by $\mathbf{x}W^3(\mathbf{x}^0)\mathbf{y} \Leftrightarrow \Pi_{i\in N}x_i \leq \Pi_{i\in N}y_i$.

[14] Even if the symmetry axiom is replaced with the non-dictatorship axiom, the impossibility theorem can be proved, by the standard manner as in Mas-Colell *et.al.* (1995, Proposition 21.C.1).

Remark 1: In Theorem 1, the symmetry axiom can be replaced with the two axioms: the weak symmetry axiom (If $\mathbf{x}^0$ satisfies $x_1^0 = x_2^0 = \cdots = x_n^0$, then $\mathbf{x}W_I(\mathbf{x}^0)\theta\circ\mathbf{x}$ for any $\mathbf{x} \in X$ and any permutation θ of N) and the anonymity axiom ($W(\mathbf{x}^0) = W(\theta\circ\mathbf{x}^0)$ for any $\mathbf{x}^0 \in X$ and any permutation θ of N).

3. The four types of aggregate consumer surplus measures and the corresponding social ordering functions in the neo-classical market model

This section introduces the four types of aggregate consumer surplus measures in the standard neo-classical (partial) market model with a finite number of consumers and a finite number of private consumption goods, and defines the Arrovian social ordering functions determined by the aggregate measures in the market model.

There are n ($n \geq 2$) number of consumers, and the set of consumers is denoted by $N = \{1, 2, \cdots, n\}$. There are m ($m \geq 2$) types of consumption goods, and the (individual) consumption set is denoted by $Y \equiv \mathbb{R}_+^m$ for all consumers. An individual preference ordering $\precsim$ is a smooth, strictly convex and strictly monotone binary relation on the consumption set Y.[15] The symmetric and asymmetric parts of $\precsim$ are denoted by $\sim$ and $\succ$, respectively. Let $\mathcal{M}$ be the set of all preference orderings on Y. An n-tuple $\boldsymbol{\succsim} \equiv (\precsim_1, \precsim_2, \cdots, \precsim_n) \in \mathcal{M}^n$ is called an N-list of individual preference orderings on Y.

An *alternative* $(\mathbf{p}, \mathbf{x})$ is a pair of a price vector $\mathbf{p} = (p_1, p_2, \cdots, p_m)$ and a positive money distribution $\mathbf{x} = (x_1, x_2, \cdots, x_n)$ on N. A *profile* is a pair of an alternative $(\mathbf{p}^0, \mathbf{x}^0)$ and an N-list of individual preference orderings $\boldsymbol{\succsim} \in \mathcal{M}^n$, where $(\mathbf{p}^0, \mathbf{x}^0)$ describes the initial state of the market, i.e., $\mathbf{p}^0$ is the initial price vector and $\mathbf{x}^0$ is the initial distribution of money.[16] For a price vector $\mathbf{p}$

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[15] More precisely, $\precsim$ is smooth, strictly convex and strictly monotone on Y's interior $\mathbb{R}_{++}^m$, and $\precsim$ is continuous on Y.

[16] A profile $(\mathbf{p}^0, \mathbf{x}^0, \boldsymbol{\succsim})$ specifies not only the basic market data $(\mathbf{x}^0, \boldsymbol{\succsim})$ as well as the initial competitive price vector $\mathbf{p}^0$ of the market before the underlying government selects a policy, and an alternative $(\mathbf{p}, \mathbf{x})$ specifies a competitive price vector $\mathbf{p}$ and the consumers' holdings of money $\mathbf{x}$ after a specific policy is implemented by the government. This implies that the money distributions as well as the price vectors are variables depending on the policies, whereas the preference orderings $\boldsymbol{\succsim}$ are invariable, whichever policy is selected.

$\in$ P and an amount of money (budget) $x > 0$, the demand function of consumer $i \in N$ with the preference ordering $\succsim_i$, $d^i(\mathbf{p}, x: \succsim_i) \in Y$, is defined by

$$\mathbf{p} \cdot d^i(\mathbf{p}, x: \succsim_i) \le x \text{ and } d^i(\mathbf{p}, x: \succsim_i) \succsim_i \mathbf{y} \text{ for all } \mathbf{y} \in Y \text{ with } \mathbf{p} \cdot \mathbf{y} \le x. \tag{5}$$

Since $d^i(\mathbf{p}, x: \succsim_i)$ is homogeneous degree zero with respect to $(\mathbf{p}, x)$ for all $i \in N$, we can normalize the price vectors by $p_1 = 1$ for any price vector $\mathbf{p} = (p_1, p_2, \cdots, p_m)$ without loss of generality. The set of alternatives is denoted by $P \times X$, where $P = \{ (p_1, p_2, \cdots, p_m) \in \mathbb{R}^m_{++} : p_1 = 1 \}$ is the set of normalized price vectors and $X \equiv \mathbb{R}^n_{++}$ is the set of positive money distributions on N. Moreover, a profile is denoted by $f = (\mathbf{p}^0, \mathbf{x}^0, \succsim)$, and the set of profiles is denoted by $P \times X \times \mathcal{M}^n$.

For any $i \in N$ and any $\mathbf{p}^0 \in P$, a pair $(x_i^0, \succsim_i) \in \mathbb{R}_{++} \times \mathcal{M}$ is called an *admissible* characteristics of consumer *i* under $\mathbf{p}^0$ if and only if $d^i(\mathbf{p}^0, x_i^0: \succsim_i)$ is a regular solution of (5), i.e., $d^i(\mathbf{p}^0, x_i^0: \succsim_i) \gg \mathbb{e}^0$, where $\mathbb{e}^0 \equiv (0, \cdots, 0)$. Let $C_i(\mathbf{p}^0) = \{ (x_i^0, \succsim_i) \in \mathbb{R}_{++} \times \mathcal{M} : d^i(\mathbf{p}^0, x_i^0: \succsim_i) \gg \mathbb{e}^0 \}$ be the set of admissible characteristics of consumer *i* under $\mathbf{p}^0$. It holds by the definition that

$$C_1(\mathbf{p}^0) = C_2(\mathbf{p}^0) = \cdots = C_n(\mathbf{p}^0) \text{ for all } \mathbf{p}^0 \in P. \tag{6}$$

Setting $P^* = \{ \mathbf{p} \in P : C_i(\mathbf{p}) \neq \emptyset \text{ for some } i \in N \}$, the set of *admissible* profiles F is defined by

$$F = \{ (\mathbf{p}^0, \mathbf{x}^0, \succsim) \in P^* \times X \times \mathcal{M}^n : (x_i^0, \succsim_i)_{i \in N} \in C_1(\mathbf{p}^0) \times C_2(\mathbf{p}^0) \times \cdots \times C_n(\mathbf{p}^0) \}. \tag{7}$$

It holds by (6) and (7) that the set of admissible profiles F is determined by the Cartesian product of the symmetric sets of the admissible individual characteristics.[17] Since the consumer *i*'s preference ordering $\succsim_i$ of the profile $f = (\mathbf{p}^0, \mathbf{x}^0, \succsim)$ completely determines the consumer *i*'s demand function for all $i \in N$, we hereafter use the notation $d^i(\mathbf{p}, x: f)$ instead of $d^i(\mathbf{p}, x: \succsim_i)$.

For any profile $f = (\mathbf{p}^0, \mathbf{x}^0, \succsim) \in F$, any price vector $\mathbf{p} \in P$ and any consumption vector $\mathbf{y} \in Y$, the consumer *i*'s expenditure function, $\mu^i(\mathbf{p}, \mathbf{y}: f)$, is defined by

$$\mu^i(\mathbf{p}, \mathbf{y}: f) = \min_{\mathbf{z} \in \{ \mathbf{w} \in Y : \mathbf{w} \succsim_i \mathbf{y} \}} \mathbf{p} \cdot \mathbf{z}. \tag{8}$$

For a given profile $f = (\mathbf{p}^0, \mathbf{x}^0, \succsim) \in F$, the consumer *i*'s *compensating variation* (CV) of an alternative $(\mathbf{p}, \mathbf{x}) \in P \times X$ is defined by

$$CV^i(\mathbf{p}, \mathbf{x}: f) = \mu^i(\mathbf{p}, d^i(\mathbf{p}, x_i: f): f) - \mu^i(\mathbf{p}, d^i(\mathbf{p}^0, x_i^0: f): f) = x^i - \mu^i(\mathbf{p}, d^i(\mathbf{p}^0, x_i^0: f): f), \tag{9}$$

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[17] In Arrow's original setting, this condition is introduced as an axiom for the social welfare functions called the unrestricted preference domain. See Le Breton and Weymark (2011, Part I, Section 1) for the axiom.

and the *equivalent variation* (EV) of $(\mathbf{p}, \mathbf{x})$ is defined by

$$EV^i(\mathbf{p}, \mathbf{x}\colon f) = \mu^i(\mathbf{p}^0, d^i(\mathbf{p}, x_i\colon f)\colon f) - \mu^i(\mathbf{p}^0, d^i(\mathbf{p}^0, x_i^0\colon f)\colon f) = \mu^i(\mathbf{p}^0, d^i(\mathbf{p}, x_i\colon f)\colon f) - x_i^0. \quad (10)$$

The two measures above are well-known. Based on Allen (1949, page 199) and Deaton (1979, Section 2), we introduce the following two measures: the consumer *i*'s *compensating ratio* (CR) of an alternative $(\mathbf{p}, \mathbf{x}) \in P{\times}X$ is defined by

$$CR^i(\mathbf{p}, \mathbf{x}\colon f) = \mu^i(\mathbf{p}, d^i(\mathbf{p}, x_i\colon f)\colon f) / \mu^i(\mathbf{p}, d^i(\mathbf{p}^0, x_i^0\colon f)\colon f) = x^i / \mu^i(\mathbf{p}, d^i(\mathbf{p}^0, x_i^0\colon f)\colon f)\colon \quad (11)$$

and the *equivalent ratio* (ER) of $(\mathbf{p}, \mathbf{x})$ is defined by

$$ER^i(\mathbf{p}, \mathbf{x}\colon f) = \mu^i(\mathbf{p}^0, d^i(\mathbf{p}, x_i\colon f)\colon f) / \mu^i(\mathbf{p}^0, d^i(\mathbf{p}^0, x_i^0\colon f)\colon f) = \mu^i(\mathbf{p}^0, d^i(\mathbf{p}, x_i\colon f)\colon f) / x_i^0. \quad (12)$$

Namely, the value of $CR^i(\mathbf{p}, \mathbf{x}\colon f)$ defined as a ratio of expenditure functions evaluated at $d^i(\mathbf{p}^0, x_i^0\colon f)$ and $d^i(\mathbf{p}, x_i\colon f)$ under a common reference price vector $\mathbf{p}$, and the value of $ER^i(\mathbf{p}, \mathbf{x}\colon f)$ defined as the same ratio under a common reference price vector $\mathbf{p}^0$.

For a given profile $f = (\mathbf{p}^0, \mathbf{x}^0, \succsim) \in F$, an alternative $(\mathbf{p}, \mathbf{x}) \in P{\times}X$ is called *admissible* iff the following regularity conditions hold:

(i) For each $i \in N$, there exists some $z_i > 0$ such that

$$d^i(\mathbf{p}^0, x_i^0\colon f) \sim_i d^i(\mathbf{p}, z_i\colon f) \text{ and } d^i(\mathbf{p}, y\colon f) \gg \mathbf{e}^0 \text{ for all } y \in [\min(x_i, z_i), \max(x_i, z_i)]\colon$$

(ii) For each $i \in N$, there exists some $z_i > 0$ such that

$$d^i(\mathbf{p}, x_i\colon f) \sim_i d^i(\mathbf{p}^0, z_i\colon f) \text{ and } d^i(\mathbf{p}^0, y\colon f) \gg \mathbf{e}^0 \text{ for all } y \in [\min(x_i^0, z_i), \max(x_i^0, z_i)].$$

Let $A(f)$ be the set of admissible alternatives of $f \in F$. Then we have the following well-known lemma:

Lemma 2: For any $f = (\mathbf{p}^0, \mathbf{x}^0, \succsim) \in F$ and any $i \in N$, the following assertions hold:

(i) $d^i(\mathbf{p}^0, x_i^0\colon f) \sim_i d^i(\mathbf{p}, x_i - CV^i(\mathbf{p}, \mathbf{x}\colon f)\colon f) \gg \mathbf{e}^0$ for all $(\mathbf{p}, \mathbf{x}) \in A(f)$.

(ii) $d^i(\mathbf{p}, x_i\colon f) \sim_i d^i(\mathbf{p}^0, x_i^0 + EV^i(\mathbf{p}, \mathbf{x}\colon f)\colon f) \gg \mathbf{e}^0$ for all $(\mathbf{p}, \mathbf{x}) \in A(f)$.

(iii) The (indirect) preference ordering of $\succsim_i$ on $A(f)$ is represented by $EV^i(\mathbf{p}, \mathbf{x}\colon f)$, i.e.,

$$d^i(\mathbf{p}, x_i\colon f) \succsim_i d^i(\mathbf{q}, y_i\colon f) \Leftrightarrow EV^i(\mathbf{p}, \mathbf{x}\colon f) \geq EV^i(\mathbf{q}, \mathbf{y}\colon f) \text{ for all } (\mathbf{p}, \mathbf{x}), (\mathbf{q}, \mathbf{y}) \in A(f).$$

(iv) $d^i(\mathbf{p}^0, x_i^0\colon f) \sim_i d^i(\mathbf{p}, x_i / CR^i(\mathbf{p}, \mathbf{x}\colon f)\colon f) \gg \mathbf{e}^0$ for all $(\mathbf{p}, \mathbf{x}) \in A(f)$.

(v) $d^i(\mathbf{p}, x_i\colon f) \sim_i d^i(\mathbf{p}^0, x_i^0 \cdot ER^i(\mathbf{p}, \mathbf{x}\colon f)\colon f) \gg \mathbf{e}^0$ for all $(\mathbf{p}, \mathbf{x}) \in A(f)$.

(vi) The (indirect) preference ordering of $\succsim_i$ on $A(f)$ is represented by $ER^i(\mathbf{p}, \mathbf{x}\colon f)$, i.e.,

$$d^i(\mathbf{p}, x_i\colon f) \succsim_i d^i(\mathbf{q}, y_i\colon f) \Leftrightarrow ER^i(\mathbf{p}, \mathbf{x}\colon f) \geq ER^i(\mathbf{q}, \mathbf{y}\colon f) \text{ for all } (\mathbf{p}, \mathbf{x}), (\mathbf{q}, \mathbf{y}) \in A(f).$$

It holds by Lemma 2(iii,vi) that $EV^i(\mathbf{p}, \mathbf{x}\colon f)$ and $ER^i(\mathbf{p}, \mathbf{x}\colon f)$ are specific forms of indirect utility functions of $\succsim_i$ on A(f). The other two measures $CV^i(\mathbf{p}, \mathbf{x}: f)$ and $CR^i(\mathbf{p}, \mathbf{x}: f)$ do not have the property as shown by Example 1 in Appendix C.

A *social ordering function* W is a function defined on F which assigns a complete and transitive binary relation on A(f) for each $f \in F$. The symmetric and asymmetric parts of W(f) are denoted by $W_I(f)$ and $W_S(f)$, respectively. Specifically, the social ordering function determined by the arithmetically aggregated compensating variations, W^{CV} is defined by

$$(\mathbf{p}, \mathbf{x})\, W^{CV}(f)\, (\mathbf{q}, \mathbf{y}) \Leftrightarrow \Sigma_{i\in N} CV^i(\mathbf{p}, \mathbf{x}: f) \geq \Sigma_{i\in N} CV^i(\mathbf{q}, \mathbf{y}: f) \tag{13}$$

for all $f \in F$ and all $(\mathbf{p}, \mathbf{x}), (\mathbf{q}, \mathbf{y}) \in A(f)$. The social ordering function W^{CV} is simply called the CV social ordering function hereafter. Replacing CV^i with EV^i in the definition of W^{CV}, the EV social ordering function, W^{EV} is defined by

$$(\mathbf{p}, \mathbf{x})\, W^{EV}(f)\, (\mathbf{q}, \mathbf{y}) \Leftrightarrow \Sigma_{i\in N} EV^i(\mathbf{p}, \mathbf{x}: f) \geq \Sigma_{i\in N} EV^i(\mathbf{q}, \mathbf{y}: f) \tag{14}$$

for all $f \in F$ and all $(\mathbf{p}, \mathbf{x}), (\mathbf{q}, \mathbf{y}) \in A(f)$.

Based on Diewert's (1984, Section 5, Equation (25)) multiplicative democratic index, the social ordering function determined by the geometrically aggregated compensating ratios, W^{CR} is defined by

$$(\mathbf{p}, \mathbf{x})\, W^{CR}(f)\, (\mathbf{q}, \mathbf{y}) \Leftrightarrow \Pi_{i\in N} CR^i(\mathbf{p}, \mathbf{x}: f) \geq \Pi_{i\in N} CR^i(\mathbf{q}, \mathbf{y}: f) \tag{15}$$

for all $f \in F$ and all $(\mathbf{p}, \mathbf{x}), (\mathbf{q}, \mathbf{y}) \in A(f)$. The social ordering function W^{CR} is called the CR social ordering function. Replacing CR^i with ER^i in the definition of W^{CR}, the ER social ordering function, W^{ER} is defined by

$$(\mathbf{p}, \mathbf{x})\, W^{ER}(f)\, (\mathbf{q}, \mathbf{y}) \Leftrightarrow \Pi_{i\in N} ER^i(\mathbf{p}, \mathbf{x}: f) \geq \Pi_{i\in N} ER^i(\mathbf{q}, \mathbf{y}: f) \tag{16}$$

for all $f \in F$ and all $(\mathbf{p}, \mathbf{x}), (\mathbf{q}, \mathbf{y}) \in A(f)$.

In order to clarify the relationships between the four social ordering functions defined above and the other normative evaluation methods in this literature, we introduce some equivalent indices each determines one of the social ordering functions above. It holds by (9) and (13) that

$$(\mathbf{p}, \mathbf{x})\, W^{CV}(f)\, (\mathbf{q}, \mathbf{y})$$

$$\Leftrightarrow \Sigma_{i\in N} x^i - \Sigma_{i\in N} \mu^i(\mathbf{p}, d^i(\mathbf{p}^0, x_i^0: f): f) \geq \Sigma_{i\in N} y^i - \Sigma_{i\in N} \mu^i(\mathbf{q}, d^i(\mathbf{p}^0, x_i^0: f): f) \tag{17}$$

for all $f \in F$ and all $(\mathbf{p}, \mathbf{x}), (\mathbf{q}, \mathbf{y}) \in A(f)$, which implies that an equivalent index of $\Sigma_{i\in N} CV^i$ is given by [arithmetically aggregated incomes] – [arithmetically aggregated (subsistence) cost-of-living]. Hence $W^{CV}(f)$ does not satisfy the Pigu-Dalton transfer principle under fixed price vectors. It holds by (11) and (15) that

$(\mathbf{p}, \mathbf{x})\ W^{CR}(f)\ (\mathbf{q}, \mathbf{y})$

$$\Leftrightarrow \Sigma_{i\in N}\log x^i - \Sigma_{i\in N}\log \mu^i(\mathbf{p}, d^i(\mathbf{p}^0, x_i^0: f): f) \geq \Sigma_{i\in N}\log y^i - \Sigma_{i\in N}\log \mu^i(\mathbf{q}, d^i(\mathbf{p}^0, x_i^0: f): f) \quad (18)$$

for all $f \in F$ and all $(\mathbf{p}, \mathbf{x}), (\mathbf{q}, \mathbf{y}) \in A(f)$, which implies that an equivalent index of $\Pi_{i\in N}CR^i$ is given by [arithmetically aggregated logarithms of incomes] – [arithmetically aggregated logarithms of (subsistence) cost-of-living]. Since the former part of this is the Atkinson index, the index can be interpreted as the price-adjusted Atkinson index. Then we have the following proposition as a direct consequence of (17), (18) and Lemma 1:

Proposition 1: For all $f \in F$ and all $(\mathbf{p}, \mathbf{x}), (\mathbf{q}, \mathbf{y}) \in A(f)$ such that $\mathbf{y}$ is obtained from $\mathbf{x}$ by a progressive transfer and $\mathbf{p} = \mathbf{q}$, it holds that $(\mathbf{q}, \mathbf{y})\ W_I^{CV}(f)\ (\mathbf{p}, \mathbf{x})$ and $(\mathbf{q}, \mathbf{y})\ W_S^{CR}(f)\ (\mathbf{p}, \mathbf{x})$.

Proposition 1 means that the social ordering $W^{CR}(f)$ satisfies the Pigu-Dalton transfer principle under fixed price vectors for all $f \in F$, although the social ordering $W^{CV}(f)$ does not satisfy the principle for all $f \in F$. In case of W^{EV} and W^{ER}, as shown in Example 2 in Appendix C, we can show that there exists a profile of market $f^* \in F$ such that the two social orderings $W^{EV}(f^*)$ and $W^{ER}(f^*)$ do not satisfy the Pigu-Dalton transfer principle for some alternatives and $W^{EV}(f^*)$ and $W^{ER}(f^*)$ satisfy the principle for some of the other alternatives.

4. Axioms for the social ordering functions in the neo-classical market model

This section attempts to extend the characterization theorem of the two social ordering functions in the simple environment of money distributions presented in Section 2 to the neo-classical market model introduced in the previous section. First, we provide some axioms in Arrovian form for the four social ordering functions in the market model, based on the three axioms in the simple environment. Concretely, the common Pareto and symmetry axioms are provided for all of the four social ordering functions, and a specific type of independence axiom is provided for each of the social ordering functions. Second, we check whether the social ordering functions can be characterized by the axioms.

Let us introduce the axioms. The Pareto axiom in the market model can be stated as:

Pareto: For any profile $f = (\mathbf{p}^0, \mathbf{x}^0, \succsim) \in F$ and any two alternatives $(\mathbf{p}, \mathbf{x}), (\mathbf{q}, \mathbf{y}) \in A(f)$,

$$d^i(\mathbf{p}, x_i: f) \succ_i d^i(\mathbf{q}, y_i: f) \text{ for all } i \in N \Rightarrow (\mathbf{p}, \mathbf{x})\ W_S(f)\ (\mathbf{q}, \mathbf{y}).$$

The Pareto axiom above is standard. We have the following lemma proved in Appendix A:

Lemma 3: **(A)** The social ordering functions W^{EV} and W^{ER} satisfy the Pareto axiom. **(B)** The social ordering functions W^{CV} and W^{CR} do not satisfy the Pareto axiom.

The next axiom is the symmetry axiom. Even though the consumers generally have different preferences (tastes) and initial money incomes in the neo-classical market model, we require the symmetry (anonymity) among the consumers in the space of the (ex post) money distributions as long as the price vector is fixed, partly since each consumer i can freely dispose i's own money income based on i's own preferences in the market.

Symmetry: $(\mathbf{p}, \mathbf{x}) W_I(f) (\mathbf{p}, \theta \circ \mathbf{x})$ for any profile $f \in F$, any alternative $(\mathbf{p}, \mathbf{x}) \in A(f)$ and any permutation θ of N such that $\theta \circ \mathbf{x} \neq \mathbf{x}$ and $(\mathbf{p}, \theta \circ \mathbf{x}) \in A(f)$, where $\theta \circ \mathbf{x} = (x_{\theta(1)}, x_{\theta(2)}, \cdots, x_{\theta(n)})$.

This axiom means the symmetry of social indifference surfaces in the space of money distributions under the fixed price vector. We have the following lemma proved in Appendix A:

Lemma 4: **(A)** The social ordering function W^{CV} and W^{CR} satisfy the symmetry axiom. **(B)** The social ordering functions W^{EV} and W^{ER} do not satisfy the symmetry axiom.

The last axiom is the independence axiom. In the neo-classical market model, the independence axiom has some variations dependent on the specification of the identical structure of the profiles. We consider the following four independence axioms:

CV-independence: For any profiles $f = (\mathbf{p}^0, \mathbf{x}^0, \succsim)$ and $f^* = (\mathbf{q}^0, \mathbf{y}^0, \succsim^*)$ in F, let $(\mathbf{p}^1, \mathbf{x}^1)$ and $(\mathbf{p}^2, \mathbf{x}^2)$ be alternatives in $A(f)$, and let $(\mathbf{q}^1, \mathbf{y}^1)$ and $(\mathbf{q}^2, \mathbf{y}^2)$ be alternatives in $A(f^*)$. Suppose that

$$d^i(\mathbf{p}^1, x_i^1+\delta_i : f) \precsim_i d^i(\mathbf{p}^0, x_i^0 : f) \Leftrightarrow d^i(\mathbf{q}^1, y_i^1+\delta_i : f^*) \precsim_i^* d^i(\mathbf{q}^0, y_i^0 : f^*)$$

holds for all $i \in N$ and all $\delta \geq -\min(x_i^1, y_i^1)$, and suppose that

$$d^i(\mathbf{p}^2, x_i^2+\delta_i : f) \precsim_i d^i(\mathbf{p}^0, x_i^0 : f) \Leftrightarrow d^i(\mathbf{q}^2, y_i^2+\delta_i : f^*) \precsim_i^* d^i(\mathbf{q}^0, y_i^0 : f^*)$$

holds for all $i \in N$ and all $\delta \geq -\min(x_i^2, y_i^2)$. Then it holds that

$$(\mathbf{p}^1, \mathbf{x}^1)\, W(f)\, (\mathbf{p}^2, \mathbf{x}^2) \Leftrightarrow (\mathbf{q}^1, \mathbf{y}^1)\, W(f^*)\, (\mathbf{q}^2, \mathbf{y}^2).$$

EV-independence: For any profiles $f = (\mathbf{p}^0, \mathbf{x}^0, \succsim)$ and $f^* = (\mathbf{q}^0, \mathbf{y}^0, \succsim^*)$ in F, let $(\mathbf{p}^1, \mathbf{x}^1)$ and $(\mathbf{p}^2, \mathbf{x}^2)$ be alternatives in $A(f)$, and let $(\mathbf{q}^1, \mathbf{y}^1)$ and $(\mathbf{q}^2, \mathbf{y}^2)$ be alternatives in $A(f^*)$. Suppose that

$d^i(\mathbf{p}^1, x^1_i : f) \precsim_i d^i(\mathbf{p}^0, x^0_i+\delta_i : f) \Leftrightarrow d^i(\mathbf{q}^1, y^1_i : f^*) \precsim^*_i d^i(\mathbf{q}^0, y^0_i+\delta_i : f^*)$ and

$d^i(\mathbf{p}^2, x^2_i : f) \precsim_i d^i(\mathbf{p}^0, x^0_i+\delta_i : f) \Leftrightarrow d^i(\mathbf{q}^2, y^2_i : f^*) \precsim^*_i d^i(\mathbf{q}^0, y^0_i+\delta_i : f^*)$

hold for all $i \in N$ and all $\delta_i \geq -\min(x^0_i, y^0_i)$. Then it holds that

$(\mathbf{p}^1, \mathbf{x}^1)\, W(f)\, (\mathbf{p}^2, \mathbf{x}^2) \Leftrightarrow (\mathbf{q}^1, \mathbf{y}^1)\, W(f^*)\, (\mathbf{q}^2, \mathbf{y}^2)$.

CR-independence: For any profiles $f = (\mathbf{p}^0, \mathbf{x}^0, \succsim)$ and $f^* = (\mathbf{q}^0, \mathbf{y}^0, \succsim^*)$ in F, let $(\mathbf{p}^1, \mathbf{x}^1)$ and $(\mathbf{p}^2, \mathbf{x}^2)$ be alternatives in A(f), and let $(\mathbf{q}^1, \mathbf{y}^1)$ and $(\mathbf{q}^2, \mathbf{y}^2)$ be alternatives in A(f*). If

$d^i(\mathbf{p}^1, \delta_i \cdot x^1_i : f) \precsim_i d^i(\mathbf{p}^0, x^0_i : f) \Leftrightarrow d^i(\mathbf{q}^1, \delta_i \cdot y^1_i : f^*) \precsim^*_i d^i(\mathbf{q}^0, y^0_i : f^*)$ and

$d^i(\mathbf{p}^2, \delta_i \cdot x^2_i : f) \precsim_i d^i(\mathbf{p}^0, x^0_i : f) \Leftrightarrow d^i(\mathbf{q}^2, \delta_i \cdot y^2_i : f^*) \precsim^*_i d^i(\mathbf{q}^0, y^0_i : f^*)$

hold for all $i \in N$ and all $\delta_i > 0$, then it holds that

$(\mathbf{p}^1, \mathbf{x}^1)\, W(f)\, (\mathbf{p}^2, \mathbf{x}^2) \Leftrightarrow (\mathbf{q}^1, \mathbf{y}^1)\, W(f^*)\, (\mathbf{q}^2, \mathbf{y}^2)$.

ER-independence: For any profiles $f = (\mathbf{p}^0, \mathbf{x}^0, \succsim)$ and $f^* = (\mathbf{q}^0, \mathbf{y}^0, \succsim^*)$ in F, let $(\mathbf{p}^1, \mathbf{x}^1)$ and $(\mathbf{p}^2, \mathbf{x}^2)$ be alternatives in A(f), and let $(\mathbf{q}^1, \mathbf{y}^1)$ and $(\mathbf{q}^2, \mathbf{y}^2)$ be alternatives in A(f*). If

$d^i(\mathbf{p}^1, x^1_i : f) \precsim_i d^i(\mathbf{p}^0, \delta_i \cdot x^0_i : f) \Leftrightarrow d^i(\mathbf{q}^1, y^1_i : f^*) \precsim^*_i d^i(\mathbf{q}^0, \delta_i \cdot y^0_i : f^*)$ and

$d^i(\mathbf{p}^2, x^2_i : f) \precsim_i d^i(\mathbf{p}^0, \delta_i \cdot x^0_i : f) \Leftrightarrow d^i(\mathbf{q}^2, y^2_i : f^*) \precsim^*_i d^i(\mathbf{q}^0, \delta_i \cdot y^0_i : f^*)$

hold for all $i \in N$ and all $\delta_i > 0$, then it holds that

$(\mathbf{p}^1, \mathbf{x}^1)\, W(f)\, (\mathbf{p}^2, \mathbf{x}^2) \Leftrightarrow (\mathbf{q}^1, \mathbf{y}^1)\, W(f^*)\, (\mathbf{q}^2, \mathbf{y}^2)$.

We have the following lemma:

Lemma 5: For any profiles $f = (\mathbf{p}^0, \mathbf{x}^0, \succsim)$ and $f^* = (\mathbf{q}^0, \mathbf{y}^0, \succsim^*)$ in F, let $(\mathbf{p}^1, \mathbf{x}^1)$ and $(\mathbf{q}^1, \mathbf{y}^1)$ be alternative in A(f) and A(f*), respectively. For any $i \in N$, the following assertions hold:

(A) $d^i(\mathbf{p}^1, x^1_i+\delta_i : f) \precsim_i d^i(\mathbf{p}^0, x^0_i : f) \Leftrightarrow d^i(\mathbf{q}^1, y^1_i+\delta_i : f^*) \precsim^*_i d^i(\mathbf{q}^0, y^0_i : f^*)$ holds for all $\delta_i \geq -\min(x^1_i, y^1_i)$ if and only if $CV^i(\mathbf{p}^1, \mathbf{x}^1 : f) = CV^i(\mathbf{q}^1, \mathbf{y}^1 : f^*)$.

(B) $d^i(\mathbf{p}^1, x^1_i : f) \precsim_i d^i(\mathbf{p}^0, x^0_i+\delta_i : f) \Leftrightarrow d^i(\mathbf{q}^1, y^1_i : f^*) \precsim^*_i d^i(\mathbf{q}^0, y^0_i+\delta_i : f^*)$ holds for all $\delta_i \geq -\min(x^0_i, y^0_i)$ if and only if $EV^i(\mathbf{p}^1, \mathbf{x}^1 : f) = EV^i(\mathbf{q}^1, \mathbf{y}^1 : f^*)$.

(C) $d^i(\mathbf{p}^1, \delta_i \cdot x^1_i : f) \precsim_i d^i(\mathbf{p}^0, x^0_i : f) \Leftrightarrow d^i(\mathbf{q}^1, \delta_i \cdot y^1_i : f^*) \precsim^*_i d^i(\mathbf{q}^0, y^0_i : f^*)$ holds for all $\delta_i > 0$ if and only if $CR^i(\mathbf{p}^1, \mathbf{x}^1 : f) = CR^i(\mathbf{q}^1, \mathbf{y}^1 : f^*)$.

(D) $d^i(\mathbf{p}^1, x^1_i : f) \precsim_i d^i(\mathbf{p}^0, \delta_i \cdot x^0_i : f) \Leftrightarrow d^i(\mathbf{q}^1, y^1_i : f^*) \precsim^*_i d^i(\mathbf{q}^0, \delta_i \cdot y^0_i : f^*)$ holds for all $\delta_i > 0$ if and only if $ER^i(\mathbf{p}^1, \mathbf{x}^1 : f) = ER^i(\mathbf{q}^1, \mathbf{y}^1 : f^*)$.

Lemma 5 is proved in Appendix A. It follows from Lemma 5 that the if-part of each independence axiom specifies the identical structure of the two profiles on the pairs of alternatives by means of the values of the corresponding individual consumer surplus measure. For example, the CV-independence axiom is equivalent to the following axiom:

CV-independence*: For any profiles f and f^* in F, let $(\mathbf{p}^1, \mathbf{x}^1)$ and $(\mathbf{p}^2, \mathbf{x}^2)$ be alternatives in $A(f)$, and let $(\mathbf{q}^1, \mathbf{y}^1)$ and $(\mathbf{q}^2, \mathbf{y}^2)$ be alternatives in $A(f^*)$. If $CV^i(\mathbf{p}^1, \mathbf{x}^1: f) = CV^i(\mathbf{q}^1, \mathbf{y}^1: f^*)$ and $CV^i(\mathbf{p}^2, \mathbf{x}^2: f) = CV^i(\mathbf{q}^2, \mathbf{y}^2: f^*)$ for all $i \in N$, then $(\mathbf{p}^1, \mathbf{x}^1)\, W(f)\, (\mathbf{p}^2, \mathbf{x}^2) \Leftrightarrow (\mathbf{q}^1, \mathbf{y}^1)\, W(f^*)\, (\mathbf{q}^2, \mathbf{y}^2)$.

Similarly, one can state the other independence axioms: EV-independence, CR-independence and ER-independence axioms in terms of individual EV^i, CR^i and ER^i values, respectively. Hence, it follows from the definition of consumer surplus measures, the CV-independence and EV-independence axioms are variants of the A-independence axiom, and the CR-independence and ER-independence axioms are variants of the R-independence axiom. Moreover, as a direct consequence of Lemma 5, we have the following lemma:

Lemma 6: The social ordering functions, W^{CV}, W^{EV}, W^{CR} and W^{ER} satisfy the CV-independence, EV-independence, CR-independence and ER-independence axioms on F, respectively.

It follows from Lemmas 3, 4 and 6 that each of the four social ordering functions does not satisfy one of the three axioms provided for the social ordering function. As a main result of this section, we show a stronger assertion that there is no social ordering function satisfying the three axioms in the neo-classical market model, whichever independence axiom is selected:

Theorem 2: (**A**) There is no social ordering function satisfying the Pareto, symmetry and CV-independence axioms on F. (**B**) There is no social ordering function satisfying the Pareto, symmetry and EV-independence axioms on F. (**C**) There is no social ordering function satisfying the Pareto, symmetry and CR-independence axioms on F. (**D**) There is no social ordering function satisfying the Pareto, symmetry and ER-independence axioms on F.

The Arrovian independence axiom in this setting can be defined as:

Arrovian independence: For any profiles $f = (\mathbf{p}^0, \mathbf{x}^0, \succsim)$ and $f^* = (\mathbf{q}^0, \mathbf{y}^0, \succsim^*)$ in F, let $(\mathbf{p}^1, \mathbf{x}^1)$ and $(\mathbf{p}^2, \mathbf{x}^2)$ be alternatives in $A(f)$, and let $(\mathbf{q}^1, \mathbf{y}^1)$ and $(\mathbf{q}^2, \mathbf{y}^2)$ be alternatives in $A(f^*)$. If

$$d^i(\mathbf{p}^1, x_i^1: f) \succsim_i d^i(\mathbf{p}^2, x_i^2: f) \Leftrightarrow d^i(\mathbf{q}^1, y_i^1: f^*) \succsim_i^* d^i(\mathbf{q}^2, y_i^2: f^*)$$

holds for all $i \in N$, then it holds that $(\mathbf{p}^1, \mathbf{x}^1)\, W(f)\, (\mathbf{p}^2, \mathbf{x}^2) \Leftrightarrow (\mathbf{q}^1, \mathbf{y}^1)\, W(f^*)\, (\mathbf{q}^2, \mathbf{y}^2)$.

Since it holds by Lemma 5(B) and Lemma 2(iii) that the if-part of the Arrovian independence axiom is weaker than the if-part of the EV independence axiom, the Arrovian independence axiom is stronger than the EV-independence axiom. Hence, it holds by Theorem 2(B) that there is no social ordering function satisfying the Pareto, symmetry and Arrovian independence axioms on F, which means that the Arrovian impossibility theorem holds in the market model.

5. Characterization of the two social ordering functions, W^{CV} and W^{EV}

This section derives the positive characterization results of the two social ordering functions W^{CV} and W^{EV}, by restricting the domains of social ordering functions. Even in case of the domains smaller than the full domain F, the domains are required to satisfy the symmetry property as the condition (6) for the full domain F, i.e., a subset of F^* of F is called an *admissible* domain if and only if there exists a subset $C_i^*(\mathbf{p}^0)$ of $C_i(\mathbf{p}^0)$ for all $i \in N$ and all $\mathbf{p}^0 \in P^* \equiv \{\mathbf{p} \in P: C_i(\mathbf{p}) \neq \emptyset$ for some $i \in N\}$ such that:

(i) $C_1^*(\mathbf{p}^0) = C_2^*(\mathbf{p}^0) = \cdots = C_n^*(\mathbf{p}^0)$ for all $\mathbf{p}^0 \in P^*$,

(ii) $F^* = \{ (\mathbf{p}^0, \mathbf{x}^0, \succsim) \in F : (x_i^0, \succsim_i)_{i\in N} \in C_1^*(\mathbf{p}^0) \times C_2^*(\mathbf{p}^0) \times \cdots \times C_n^*(\mathbf{p}^0) \}$.

In particular, we consider the quasi-linear domain in which all individual preference orderings are quasi-linear. Formally, a profile $f \in F$ is *quasi-linear* if and only if for all $i \in N$ and all $(\mathbf{p}, \mathbf{x}), (\mathbf{q}, \mathbf{y}) \in A(f)$

$$d^i(\mathbf{p}, x_i: f) \sim_i d^i(\mathbf{q}, y_i: f) \Rightarrow d^i(\mathbf{p}, x_i: f) + \delta\cdot\mathbf{e}^1 \sim_i d^i(\mathbf{q}, y_i: f) + \delta\cdot\mathbf{e}^1 \text{ for all } \delta > 0.$$

Let F^L be the set of quasi-linear profiles. The set F^L has the following properties:

Lemma 7: **(A)** F^L is an admissible domain. **(B)** For any $f = (\mathbf{p}^0, \mathbf{x}^0, \succsim) \in F$, the following statements are mutually equivalent:

(i) $f \in F^L$.

(ii) $d^i(\mathbf{p}, x_i+\delta: f) = d^i(\mathbf{p}, x_i: f) + \delta\cdot\mathbf{e}^1$ for all $i \in N$, all $(\mathbf{p}, \mathbf{x}) \in A(f)$ and all $\delta > 0$.

(iii) $CV^i(\mathbf{p}, \mathbf{x}: f) = EV^i(\mathbf{p}, \mathbf{x}: f)$ for all $i \in N$ and all $(\mathbf{p}, \mathbf{x}) \in A(f)$.

(iv) The indirect preference ordering of $\succsim_i$ on $A(f)$ is represented by $CV^i(\mathbf{p}, \mathbf{x}: f)$ for all $i \in N$, i.e., $d^i(\mathbf{p}, x_i: f) \succsim_i d^i(\mathbf{q}, y_i: f) \Leftrightarrow CV^i(\mathbf{p}, \mathbf{x}: f) \geq CV^i(\mathbf{q}, \mathbf{y}: f)$ for all $(\mathbf{p}, \mathbf{x}), (\mathbf{q}, \mathbf{y}) \in A(f)$.

(C) Let F^* be an admissible domain such that $F^L \subset F^*$ and $F^* \neq F^L$. There exists a profile $g \in F^*$ on which W^{CV} does not satisfy the Pareto axiom, and there exists a profile $h \in F^*$ on which W^{EV} does not satisfy the symmetry axiom. **(D)** For any profiles $f = (\mathbf{p}^0, \mathbf{x}^0, \succsim)$, $g = (\mathbf{q}^0, \mathbf{y}^0, \succsim^*) \in F^L$, if $\succsim = \succsim^*$, then $W^{CV}(f) = W^{CV}(g)$ on $A(f) \cap A(g)$.

Lemma 7 is proved in Appendix A. It follows from Lemma 7(A) that F^L is determined by the Cartesian product of the symmetric sets of individual characteristics satisfying the quasi-linear condition. Lemma 7B(i $\Leftrightarrow$ ii) means that our quasi-linearity condition is equivalent to the standard definition of the quasi-linearity in terms of preference orderings on the consumption set, i.e., the parallelness of indifference surfaces, and Lemma 7B(i $\Leftrightarrow$ iii) means that the W^{CV} social ordering function coincides with the W^{EV} social ordering function on F^L. Consequently, the following proposition holds by Proposition 1:

Proposition 2: For all $f \in F^L$ and all $(\mathbf{p}, \mathbf{x}), (\mathbf{q}, \mathbf{y}) \in A(f)$ such that $\mathbf{y}$ is obtained from $\mathbf{x}$ by a progressive transfer and $\mathbf{p} = \mathbf{q}$, it holds that $(\mathbf{p}, \mathbf{y})\ W_I^{EV}(f)\ (\mathbf{q}, \mathbf{x})$.

Proposition 2 means that the social ordering $W^{EV}(f)$ does not satisfy the Pigu-Dalton transfer principle under fixed price vectors for all quasi-linear profiles $f \in F^L$. Moreover, it holds by Lemma 7B(i $\Leftrightarrow$ iii) that the CV-independence axiom is weaker than the Arrovian independence axiom on F^L. It holds by Lemma 7(B,D) that the two social orderings $W^{CV}(f)$ and $W^{EV}(f)$ are independent of the selections of the initial states $(\mathbf{p}^0, \mathbf{x}^0)$ on the quasi-linear profiles $f = (\mathbf{p}^0, \mathbf{x}^0, \succsim) \in F^L$.

The Lemmas 3(A), 4(A), 6 and 7B(i $\Leftrightarrow$ iii) together imply that the two social ordering functions, W^{CV} and W^{EV} satisfy the Pareto, symmetry, CV-independence and EV-independence axioms on F^L. The next theorem implies that there is no other social ordering function satisfying the axioms on the domain:

Theorem 3: **(A)** The following four statements for a social ordering function W defined on the quasi-linear domain F^L are mutually equivalent:

(i) W satisfies the Pareto, symmetry and CV-independence axioms,

(ii) W satisfies the Pareto, symmetry, and EV-independence axioms,

(iii) W coincides with W^{CV}, i.e.,

$$(\mathbf{p}, \mathbf{x})\ W(f)\ (\mathbf{q}, \mathbf{y}) \Leftrightarrow \textstyle\sum_{i \in N} CV^i(\mathbf{p}, \mathbf{x}: f) \geq \sum_{i \in N} CV^i(\mathbf{q}, \mathbf{y}: f) \text{ for all } f \in F^L \text{ and all } (\mathbf{p}, \mathbf{x}), (\mathbf{q}, \mathbf{y}) \in A(f),$$

(iv) W coincides with W^{EV}, i.e.,

$$(\mathbf{p}, \mathbf{x})\, W(f)\, (\mathbf{q}, \mathbf{y}) \Leftrightarrow \Sigma_{i\in N} EV^i(\mathbf{p}, \mathbf{x}: f) \geq \Sigma_{i\in N} EV^i(\mathbf{q}, \mathbf{y}: f) \text{ for all } f \in F^L \text{ and all } (\mathbf{p}, \mathbf{x}), (\mathbf{q}, \mathbf{y}) \in A(f).$$

(B) The quasi-linear domain F^L is the maximal admissible domain for the equivalence of the statements (i) and (iii) in Theorem 3(A) (i.e., if the equivalence holds on an admissible domain F^* such that $F^L \subset F^*$, then $F^* = F^L$) and for the equivalence of the statements (ii) and (iv) in Theorem 3(A). **(C)** There is no social ordering function satisfying the Pareto, symmetry and Arrovian independence axioms on F^L.

Theorem 3(A) clarifies the underlying principles characterizing the two aggregate consumer surplus measures, $\Sigma_{i\in N} CV^i$ and $\Sigma_{i\in N} EV^i$ as the social welfare indices, and tell us that the measures have solid normative foundations in the neoclassical market model, even when both the price vector and the money distribution are variable. Theorem 3(B) implies that it is necessary to restrict the domain to the quasi-linear one for each of the characterization results of Theorem 3(A) to hold, and Theorem 3(C) implies that it is necessary to weaken the independence axiom for the characterization result of Theorem 3(A) to hold. Hence the Theorem 3(B, C) together imply that both of the two modifications: one is restriction of the domain and the other is weakening the independence axiom are necessary to escape from the Arrovian impossibility.

The three axioms in the assertion (i) of Theorem 3(A) are mutually independent on F^L, and the three axioms in the assertion (ii) of Theorem 3(A) are mutually independent on F^L. Since it holds by Lemma 7(B) and Lemma 5(A, B) that a social ordering function W satisfies the CV-independence axiom on F^L if and only if W satisfies the EV-independence axiom on F^L, it suffices to prove that the three axioms in the assertion (ii) are mutually independent, which can be proved by constructing the counter examples as for the independence of the axioms in Theorem 1(A). Concretely, define the three social ordering functions, H^1, H^2 and H^3 on F^L by

$$(\mathbf{p}, \mathbf{x})\, H^1(f)\, (\mathbf{q}, \mathbf{y}) \Leftrightarrow \Sigma_{i\in N} EV^i(\mathbf{p}, \mathbf{x}: f) \leq \Sigma_{i\in N} EV^i(\mathbf{q}, \mathbf{y}: f),$$

$$(\mathbf{p}, \mathbf{x})\, H^2(f)\, (\mathbf{q}, \mathbf{y}) \Leftrightarrow \Pi_{i\in N} EV^i(\mathbf{p}, \mathbf{x}: f) \geq \Pi_{i\in N} EV^i(\mathbf{q}, \mathbf{y}: f),$$

$$(\mathbf{p}, \mathbf{x})\, H^3(f)\, (\mathbf{q}, \mathbf{y}) \Leftrightarrow \begin{cases} (\mathbf{p}, \mathbf{x})\, H^2(f)\, (\mathbf{q}, \mathbf{y}) & \text{if } x_i^0 = x_j^0 \text{ and } \succsim_i = \succsim_j \text{ for all } i, j \in N \\ (\mathbf{p}, \mathbf{x})\, W^{EV}(f)\, (\mathbf{q}, \mathbf{y}) & \text{otherwise} \end{cases}$$

for all $f = (\mathbf{p}^0, \mathbf{x}^0, \succsim) \in F^L$ and all $(\mathbf{p}, \mathbf{x}), (\mathbf{q}, \mathbf{y}) \in A(f)$. It holds that H^3 satisfies the Pareto and symmetry axioms on F^L, but H^3 does not satisfy the EV-independence axiom on F^L, which implies that the EV-independence axiom is independent of the other axioms. Similarly, we can prove the independence of the Pareto and symmetry axioms, making use of H^1 and H^2, respectively.

6. Characterization of the two social ordering functions, W^{CR} and W^{ER}

This section derives the positive characterization results of the two social ordering functions, W^{CR} and W^{ER}, by restricting the domain to the homothetic domain in which all individual preference orderings are homothetic. Formally, a profile $f \in F$ is *homothetic* if and only if for all $i \in N$ and all $(\mathbf{p}, \mathbf{x}), (\mathbf{q}, \mathbf{y}) \in A(f)$

$$d^i(\mathbf{p}, x_i: f) \sim_i d^i(\mathbf{q}, y_i: f) \Rightarrow \delta \cdot d^i(\mathbf{p}, x_i: f) \sim_i \delta \cdot d^i(\mathbf{q}, y_i: f) \quad \text{for all } \delta > 0.$$

Let F^H be the set of homothetic profiles.[18] The set F^H has the following properties:

Lemma 8: **(A)** F^H is an admissible domain. **(B)** For any $f = (\mathbf{p}^0, \mathbf{x}^0, \succsim) \in F$, the following statements are mutually equivalent:

(i) $f \in F^H$.

(ii) $d^i(\mathbf{p}, \delta \cdot x_i: f) = \delta \cdot d^i(\mathbf{p}, x_i: f)$ for all $i \in N$, all $(\mathbf{p}, \mathbf{x}) \in A(f)$ and all $\delta > 0$.

(iii) $CR^i(\mathbf{p}, \mathbf{x}: f) = ER^i(\mathbf{p}, \mathbf{x}: f)$ for all $i \in N$ and all $(\mathbf{p}, \mathbf{x}) \in A(f)$.

(iv) The indirect preference ordering of $\succsim_i$ on $A(f)$ is represented by $CR^i(\mathbf{p}, \mathbf{x}: f)$ for all $i \in N$, i.e., $d^i(\mathbf{p}, x_i: f) \succsim_i d^i(\mathbf{q}, y_i: f) \Leftrightarrow CR^i(\mathbf{p}, \mathbf{x}: f) \geq CR^i(\mathbf{q}, \mathbf{y}: f)$ for all $(\mathbf{p}, \mathbf{x}), (\mathbf{q}, \mathbf{y}) \in A(f)$.

(C) Let F^* be an admissible domain such that $F^H \subset F^*$ and $F^* \neq F^H$. There exists a profile $g \in F^*$ on which W^{CR} does not satisfy the Pareto axiom, and there exists a profile $h \in F^*$ on which W^{ER} does not satisfy the symmetry axiom. **(D)** For any profiles $f = (\mathbf{p}^0, \mathbf{x}^0, \succsim)$, $g = (\mathbf{q}^0, \mathbf{y}^0, \succsim^*) \in F^H$, if $\succsim = \succsim^*$, then $A(f) = A(g)$ and $W^{CR}(f) = W^{CR}(g)$ on $A(f)$.

Lemma 8 is proved in Appendix A. It follows from Lemma 8(A) that F^H is determined by the

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18 The homothetic domain is needed for normative evaluation methods in the different approaches: the representative consumer theory as in Mas-Colell, *et.al.* (1995, Section 4) and Blackorby and Donaldson's (1987) cost-benefit rule. However, they show that some additional conditions are needed. In case of the representative consumer theory, Mas-Colell, *et.al.* (1995, Exercise 3.G.12, Examples 4.D.2 and 4.D.1) introduce the condition that all the consumers have the same homothetic preferences. In case of the cost-benefit rule, Blackorby and Donaldson (1987, Theorem 6) introduce an additional interpersonal condition called the fully homothetic condition. See also Slesnick (1998, Section 3.1) and the references.

Cartesian product of the symmetric sets of individual characteristics satisfying the homotheticity condition. Lemma 8B(i $\Leftrightarrow$ ii) means that our homotheticity condition is equivalent to the standard definition of the homotheticity in terms of preference orderings on the consumption set, and Lemma 8B(i $\Leftrightarrow$ iii) means that the W^{CR} social ordering function coincides with the W^{ER} social ordering function on F^H. Consequently, we have by Proposition 1 that the following proposition holds:

Proposition 3: For all $f \in F^H$ and all $(\mathbf{p}, \mathbf{x}), (\mathbf{q}, \mathbf{y}) \in A(f)$ such that $\mathbf{y}$ is obtained from $\mathbf{x}$ by a progressive transfer and $\mathbf{p} = \mathbf{q}$, it holds that $(\mathbf{q}, \mathbf{y})\, W_S^{ER}(f)\, (\mathbf{p}, \mathbf{x})$.

Proposition 3 means that $W^{ER}(f)$ satisfies the Pigu-Dalton transfer principle under fixed price vectors for all homothetic profiles $f \in F^H$. Moreover, it holds by Lemma 8B(i $\Leftrightarrow$ iii) that the CR-independence axiom is weaker than the Arrovian independence axiom. It holds by Lemma 7(B,D) that the two social orderings $W^{CR}(f)$ and $W^{ER}(f)$ are independent of the selections of the initial states $(\mathbf{p}^0, \mathbf{x}^0)$ on the homothetic profiles $f = (\mathbf{p}^0, \mathbf{x}^0, \succsim) \in F^H$. In case of the Cobb-Douglas type utility functions, which is a typical case of homothetic preferences, the two social ordering functions is determined by the simple indices. For example, set the Cobb-Douglas type utility function U^i by $U^i(z_1, z_2, \cdots, z_m) = z_1^{\alpha_1(i)} \cdot z_2^{\alpha_2(i)} \cdot \cdots \cdot z_m^{\alpha_m(i)}$ for all $i \in N$, where $\alpha_k(i)$ is consumer *i*'s budget share of k-good, i.e., $\alpha_k(i) > 0$ for all k and $\Sigma_k \alpha_k(i) = 1$. Letting $\mathbf{r}$ and $\mathbf{z}$ be any two vectors in P and X, respectively, define a profile *f* by $f = (\mathbf{r}, \mathbf{z}, \succsim)$, where $\succsim \in \mathcal{M}^n$ is defined by $\mathbf{x} \succsim_i \mathbf{y} \Leftrightarrow U^i(\mathbf{x}) \geq U^i(\mathbf{y})$ for all $i \in N$. The it holds that

$$
\begin{aligned}
(\mathbf{p}, \mathbf{x})\, W^{CR}(f)\, (\mathbf{q}, \mathbf{y}) \;&\Leftrightarrow\; (\mathbf{p}, \mathbf{x})\, W^{ER}(f)\, (\mathbf{q}, \mathbf{y}) \\
&\Leftrightarrow\; \Pi_{i \in N}\, x_i/[\, p_1^{\alpha_1(i)} \cdot p_2^{\alpha_2(i)} \cdot \cdots \cdot p_m^{\alpha_m(i)}\,] \cdot [\, r_1^{\alpha_1(i)} \cdot r_2^{\alpha_2(i)} \cdot \cdots \cdot r_m^{\alpha_m(i)}\,] \\
&\qquad\qquad \geq\; \Pi_{i \in N}\, y_i/[\, q_1^{\alpha_1(i)} \cdot q_2^{\alpha_2(i)} \cdot \cdots \cdot q_m^{\alpha_m(i)}\,] \cdot [\, r_1^{\alpha_1(i)} \cdot r_2^{\alpha_2(i)} \cdot \cdots \cdot r_m^{\alpha_m(i)}\,] \\
&\Leftrightarrow\; \Pi_{i \in N}\, x_i/p_1^{\alpha_1(i)} \cdot p_2^{\alpha_2(i)} \cdot \cdots \cdot p_m^{\alpha_m(i)} \geq \Pi_{i \in N}\, y_i/q_1^{\alpha_1(i)} \cdot q_2^{\alpha_2(i)} \cdot \cdots \cdot q_m^{\alpha_m(i)} \\
&\Leftrightarrow\; \Sigma_{i \in N} \log x_i - \Sigma_k \alpha_k \log p_k \geq \Sigma_{i \in N} \log y_i - \Sigma_k \alpha_k \log q_k \quad \text{for all } (\mathbf{p}, \mathbf{x}), (\mathbf{q}, \mathbf{y}) \in A(f),
\end{aligned}
$$

where α_k is the number defined by $\alpha_k = \Sigma_{i \in N} \alpha_k(i)$ for all *k*.[19]

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[19] Since the number α_k is derived by directly aggregating the individual budget share of the *k*-th good, α_k is generally different from the aggregate budget share of the *k*-th good in the literature of consumer demand analysis, which is defined by the budget share of the *k*-th good of the aggregate demand. See Blundell and Stoker (2007, Section 2.1.1) for the aggregate budget share.

The Lemmas 3(A), 4(A), 6 and 8B(i $\Leftrightarrow$ iii) together imply that the two social ordering functions, W^{CR} and W^{ER} satisfy the Pareto, symmetry, CR-independence and ER-independence axioms on F^H. The next theorem implies that there is no other social ordering function satisfying the axioms on the domain:

Theorem 4: **(A)** The following four statements for a social ordering function W defined on the homothetic domain F^H are mutually equivalent:

(i) W satisfies the Pareto, symmetry and CR-independence axioms,

(ii) W satisfies the Pareto, symmetry and ER-independence axioms,

(iii) W coincides with W^{CR}, i.e.,

$$(\mathbf{p}, \mathbf{x})\ W(f)\ (\mathbf{q}, \mathbf{y}) \Leftrightarrow \Pi_{i\in N}CR^i(\mathbf{p}, \mathbf{x}: f) \geq \Pi_{i\in N}CR^i(\mathbf{q}, \mathbf{y}: f) \text{ for all } f \in F^H \text{ and all } (\mathbf{p}, \mathbf{x}), (\mathbf{q}, \mathbf{y}) \in A(f),$$

(iv) W coincides with W^{ER}, i.e.,

$$(\mathbf{p}, \mathbf{x})\ W(f)\ (\mathbf{q}, \mathbf{y}) \Leftrightarrow \Pi_{i\in N}ER^i(\mathbf{p}, \mathbf{x}: f) \geq \Pi_{i\in N}ER^i(\mathbf{q}, \mathbf{y}: f) \text{ for all } f \in F^H \text{ and all } (\mathbf{p}, \mathbf{x}), (\mathbf{q}, \mathbf{y}) \in A(f).$$

(B) The homothetic domain F^H is the maximal admissible domain for the equivalence of the statements (i) and (iii) in Theorem 4(A) and for the equivalence of the statements (ii) and (iv) in Theorem 4(A). **(C)** There is no social ordering function satisfying the Pareto, symmetry and Arrovian independence axioms on F^H.

Theorem 4(A) clarifies the underlying principles characterizing the two aggregate consumer surplus measures, $\Pi_{i\in N}CR^i$ and $\Pi_{i\in N}ER^i$ as the social welfare indices, and tell us that the measures have solid normative foundations in the neoclassical environment. Theorem 4(B) implies that it is necessary to restrict the domain into the homothetic one for each of the characterization results of Theorem 4(A) to hold, and Theorem 4(C) implies that it is necessary to weaken the independence axiom for the characterization result of Theorem 4(A) to hold. Hence the Theorem 4(B, C) together imply that both of the two modifications: one is restriction of the domain and the other is weakening the independence axiom are necessary to escape from the Arrovian impossibility. Moreover, it holds by Propositions 1 and 4 that Π_iCR^i and Π_iER^i are equity-regarding social welfare indices on the homothetic domain, but it holds by Propositions 1 and 3 that Σ_iCV^i and Σ_iEV^i are not equity-regarding on the quasi-linear domain. Comparing Theorems 3(A) and 4(A), it can be concluded that the equity-regarding property comes from the CR-independence and ER-independence axioms.

The three axioms in the assertion (i) of Theorem 4(A) are mutually independent on F^H and the three axioms in the assertion (ii) of Theorem 4(A) are mutually independent on F^H, which can be proved by constructing the counter examples as for the independence of the axioms in Theorem 3(A). Concretely, define the three social ordering functions, L^1, L^2 and L^3 on F^H by

$$(\mathbf{p}, \mathbf{x})\, L^1(f)\, (\mathbf{q}, \mathbf{y}) \;\Leftrightarrow\; \Pi_{i\in N} ER^i(\mathbf{p}, \mathbf{x}: f) \le \Pi_{i\in N} ER^i(\mathbf{q}, \mathbf{y}: f),$$

$$(\mathbf{p}, \mathbf{x})\, L^2(f)\, (\mathbf{q}, \mathbf{y}) \;\Leftrightarrow\; \Sigma_{i\in N} ER^i(\mathbf{p}, \mathbf{x}: f) \ge \Sigma_{i\in N} ER^i(\mathbf{q}, \mathbf{y}: f),$$

$$(\mathbf{p}, \mathbf{x})\, L^3(f)\, (\mathbf{q}, \mathbf{y}) \;\Leftrightarrow\; \begin{cases} (\mathbf{p}, \mathbf{x})\, L^2(f)\, (\mathbf{q}, \mathbf{y}) & \text{if } x_i^0 = x_j^0 \text{ and } \succsim_i = \succsim_j \text{ for all } i, j \in N \\ (\mathbf{p}, \mathbf{x})\, W^{ER}(f)\, (\mathbf{q}, \mathbf{y}) & \text{otherwise} \end{cases}$$

for all $f = (\mathbf{p}^0, \mathbf{x}^0, \succsim) \in F^H$ and all $(\mathbf{p}, \mathbf{x}), (\mathbf{q}, \mathbf{y}) \in A(f)$. It holds that L^3 satisfies the Pareto and symmetry axioms on F^H, but L^3 does not satisfy the ER-independence axiom on F^H, which implies that the ER-independence axiom is independent of the other axioms. Similarly, we can prove the independence of the Pareto and symmetry axioms, making use of L^1 and L^2, respectively.

7. Proof of Theorems

This section proves Theorems 1, 3, 4 and 2 in this order. We need some concepts to prove Theorem 1. Let $\mathrm{Log}: X \to X$ and $\mathrm{Exp}: \mathbb{R}^n \to X$ be the vector-valued functions defined by

$$\mathrm{Log}(\mathbf{x}) = (\log x_1, \log x_2, \cdots, \log x_n) \text{ and } \mathrm{Exp}(\mathbf{x}) = (e^{x_1}, e^{x_2}, \cdots, e^{x_n}) \text{ for all } \mathbf{x} = (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n,$$

and denote

$$\mathbf{x} * \mathbf{y} = (x_1 \cdot y_1, x_2 \cdot y_2, \cdots, x_n \cdot y_n) \in X \text{ for all } \mathbf{x} = (x_1, x_2, \cdots, x_n), \mathbf{y} = (y_1, y_2, \cdots, y_n) \in X.$$

Proof of Theorem 1: First, we prove Theorem 1(B). **(B)** We can easily show that W^G satisfies the Pareto, symmetry and R-independence axioms. Conversely, suppose that a social ordering function W satisfies the three axioms. We need the following lemma:

Lemma 9: **(i)** $\mathbf{a} W(\omega) \mathbf{b} \Leftrightarrow (\mathbf{a}*\mathbf{c}) W(\omega*\mathbf{c}) (\mathbf{b}*\mathbf{c})$ for all $\omega, \mathbf{a}, \mathbf{b}, \mathbf{c} \in X$. **(ii)** For any ordered pair $\mathbf{a}$, $\mathbf{b} \in \mathbb{R}^n$, it holds that $\mathrm{Exp}(\mathbf{a}) W(\mathbf{e}) \mathrm{Exp}(\mathbf{b}) \Leftrightarrow \Sigma_{i\in N} a_i \ge \Sigma_{i\in N} b_i$, where $\mathbf{e} = (1, 1, \ldots, 1)$.

For given $\mathbf{x}^0, \mathbf{x}, \mathbf{y} \in X$, define $\mathbf{a}$ and $\mathbf{b}$ in $\mathbb{R}^n$ by $\mathbf{a} = \mathrm{Log}(\mathbf{x}/\mathbf{x}^0)$ and $\mathbf{b} = \mathrm{Log}(\mathbf{y}/\mathbf{x}^0)$, where $\mathbf{x}/\mathbf{x}^0$ is defined in Section 2. We have by (4) and Lemma 9(ii, i) that

$$\begin{aligned}
\mathbf{x} W^G(\mathbf{x}^0) \mathbf{y} &\Leftrightarrow \Pi_{i\in N} x_i \ge \Pi_{i\in N} y_i \Leftrightarrow \Pi_{i\in N} x_i / \Pi_{i\in N} x_i^0 \ge \Pi_{i\in N} y_i / \Pi_{i\in N} x_i^0 \\
&\Leftrightarrow \Sigma_{i\in N} \log(x_i/x_i^0) \ge \Sigma_{i\in N} \log(y_i/x_i^0) \Leftrightarrow \Sigma_{i\in N} a_i \ge \Sigma_{i\in N} b_i \Leftrightarrow \mathbf{x}/\mathbf{x}^0\, W(\mathbf{e})\, \mathbf{y}/\mathbf{x}^0 \\
&\Leftrightarrow (\mathbf{x}/\mathbf{x}^0)*\mathbf{x}^0\, W(\mathbf{e}*\mathbf{x}^0)\, (\mathbf{y}/\mathbf{x}^0)*\mathbf{x}^0 \Leftrightarrow \mathbf{x} W(\mathbf{x}^0) \mathbf{y},
\end{aligned}$$

which implies that W coincides with W^G. **(A)** We can easily show that W^A satisfies the Pareto, symmetry and A-independence axioms. Conversely, suppose that a social ordering function W on X satisfies the three axioms. Define an extended social ordering function W^* of W on $\mathbb{R}^n$ by

$$\mathbf{x}W^*(\mathbf{x}^0)\mathbf{y} \Leftrightarrow (\mathbf{x}+\mathbf{w})\, W(\mathbf{x}^0+\mathbf{w})\, (\mathbf{y}+\mathbf{w}) \quad \text{for any } \mathbf{x}^0, \mathbf{x}, \mathbf{y} \in \mathbb{R}^n.$$

where $\mathbf{w} = w \cdot \mathbf{e}$ and $w = \max[\, \max_{i\in N}|x_i|, \max_{i\in N}|y_i|, \max_{i\in N}|x_i^0|\,] + 1$. We can prove easily that W^* satisfies the following conditions:

$$\mathbf{x}W(\mathbf{x}^0)\mathbf{y} \Leftrightarrow \mathbf{x}W^*(\mathbf{x}^0)\mathbf{y} \quad \text{for any } \mathbf{x}^0, \mathbf{x}, \mathbf{y} \in X; \tag{19}$$

W^* satisfies the Pareto, symmetry and A-independence axioms on $\mathbb{R}^n$. (20)

Define a social ordering function T on X by

$$\mathbf{x}T(\mathbf{x}^0)\mathbf{y} \Leftrightarrow \mathrm{Log}(\mathbf{x})\, W^*(\mathrm{Log}(\mathbf{x}^0))\, \mathrm{Log}(\mathbf{y}) \quad \text{for any } \mathbf{x}^0, \mathbf{x}, \mathbf{y} \in X. \tag{21}$$

Then it holds by (20) that T satisfies the Pareto, symmetry and R-independence axioms on X. Thus it holds by Theorem 1(B) that T coincides with W^G on X, and then it holds by (21) and (4) that

$$\mathbf{x}W^*(\mathbf{x}^0)\mathbf{y} \Leftrightarrow \mathrm{Exp}(\mathbf{x})\, T(\mathrm{Exp}(\mathbf{x}^0))\, \mathrm{Exp}(\mathbf{y}) \Leftrightarrow \Pi_{i\in N} e^{x_i} \geq \Pi_{i\in N} e^{y_i}$$

$$\Leftrightarrow \Sigma_{i\in N} x_i \geq \Sigma_{i\in N} y_i \quad \text{for any } \mathbf{x}^0, \mathbf{x}, \mathbf{y} \in X.$$

Thus we have by (19) and (2) that $\mathbf{x}W(\mathbf{x}^0)\mathbf{y} \Leftrightarrow \mathbf{x}W^*(\mathbf{x}^0)\mathbf{y} \Leftrightarrow \Sigma_{i\in N} x_i \geq \Sigma_{i\in N} y_i \Leftrightarrow \mathbf{x}W^A(\mathbf{x}^0)\mathbf{y}$ for any $\mathbf{x}^0$, $\mathbf{x}, \mathbf{y} \in X$, which implies that W coincides with W^A. □

Proof of Theorem 3: **(A)** It holds by Lemma 7B(i ⇔ iii) that

$$W^{CV} = W^{EV} \text{ on } F^L, \tag{22}$$

which implies that the statements (iii) and (iv) are equivalent. Moreover, it holds by Lemma 7B(i ⇔ iii) and Lemma 5(A, B) that W satisfies the CV-independence axiom on F^L iff W satisfies the EV-independence axiom on F^L, which implies that the statements (i) and (ii) are equivalent. There remains to show that the statements (i) and (iii) are equivalent. Suppose that the statement (iii) holds. It holds by Lemma 4(A) and Lemma 6 that W^{CV} satisfies the symmetry and CV-independence axioms, and it holds by (22) and Lemma 3(A) that W^{CV} satisfies the Pareto axiom on F^L, which implies that the statement (i) holds. Conversely, suppose that the statement (i) holds. We need a lemma:

Lemma 10: If a social ordering function W satisfies the Pareto, symmetry and CV-independence axioms on F^L, then W coincides with W^{CV} on F^L.

This lemma implies that the statement (iii) holds. **(B)** Suppose that the equivalence of (i) and (iii) in Theorem 3(A) holds on some admissible domain F^* such that $F^L \subset F^*$ and that $F^L \neq F^*$. It holds by Lemma 7(C) that exists some $g \in F^*$ on which W^{CV} does not satisfy the Pareto axiom, which contradicts the supposition. Similarly, we can prove the maximality of F^L for the characterization of W^{EV}. **(C)** Suppose that there is a social ordering function W satisfying the Pareto, symmetry and Arrovian independence axioms on F^L. Since it holds by Lemma 5(B) and Lemma 2(iii) that the Arrovian independence axiom is stronger than EV-independence axiom, it holds by Theorem 3(B) that $W = W^{EV}$. This is a contradiction, since W^{EV} does not satisfy the Arrovian independence axiom on F^L. □

Proof of Theorem 4: **(A)** It holds by Lemma 8B(i $\Leftrightarrow$ iii) that

$$W^{CR} = W^{ER} \text{ on } F^H. \tag{23}$$

which implies that the statements (iii) and (iv) are equivalent. Moreover, it holds by Lemma 8B(i $\Leftrightarrow$ iii) and Lemma 5(C, D) that W satisfies the CR-independence axiom on F^H iff W satisfies the ER-independence axiom on F^H, which implies that the statements (i) and (ii) are equivalent. There remains to show that the statements (i) and (iii) are equivalent. Suppose that the statement (iii) holds. It holds by Lemma 4(A) and Lemma 6 that W^{CR} satisfies the symmetry and CR-independence axioms, and it holds by (23) and Lemma 3(A) that W^{CR} satisfies the Pareto axiom on F^H, which implies that the statement (i) holds. Conversely, suppose that the statement (i) holds. We need a lemma.

Lemma 11: If a social ordering function W satisfies the Pareto, symmetry and CR-independence axioms on F^H, then W coincides with W^{CR} on F^H.

This lemma implies that the statement (iii) holds. **(B)** Suppose that the equivalence of (i) and (iii) in Theorem 4(A) holds on some admissible domain F^* such that $F^H \subset F^*$ and $F^H \neq F^*$. It holds by Lemma 8(C) that exists some $g \in F^*$ on which W^{CR} does not satisfy the Pareto axiom, which contradicts the supposition. Similarly, we can prove the maximality of F^H for the characterization of W^{ER}. **(C)** Suppose that there is a social ordering function W satisfying the Pareto, symmetry and Arrovian independence axioms on F^H. Since it holds by Lemma 5(D) and Lemma 2(vi) that the Arrovian independence axiom is stronger than ER-independence axiom, it holds by Theorem 4(B) that $W = W^{ER}$ on F^H. This is a contradiction, since W^{ER} does not satisfy the Arrovian independence axiom on F^H. □

Proof of Theorem 2: All the assertions in Theorem 2 are direct consequences of Theorem 3(B) and Theorem 4(B). □

Appendix A

Proof of Lemma 2: **(i)** It holds by (9) that $x_i - CV^i(\mathbf{p}, \mathbf{x}: f) = \mu^i(\mathbf{p}, d^i(\mathbf{p}^0, x_i^0: f): f)$ for any $(\mathbf{p}, \mathbf{x}) \in A(f)$, which implies that $d^i(\mathbf{p}, x_i - CV^i(\mathbf{p}, \mathbf{x}: f): f) = d^i(\mathbf{p}, \mu^i(\mathbf{p}, d^i(\mathbf{p}^0, x_i^0: f): f): f)$. Moreover, it holds by (8) that $d^i(\mathbf{p}, \mu^i(\mathbf{p}, d^i(\mathbf{p}^0, x_i^0: f): f): f) \sim_i d^i(\mathbf{p}^0, x_i^0: f)$. Hence, it holds by the condition (i) of the admissibility of alternatives that $d^i(\mathbf{p}, x_i - CV^i(\mathbf{p}, \mathbf{x}: g): g) \gg \mathbf{e}^0$. **(ii)** It holds by (10) and (8) that

$$d^i(\mathbf{p}^0, x_i^0 + EV^i(\mathbf{p}, \mathbf{x}: f): f) = d^i(\mathbf{p}^0, \mu^i(\mathbf{p}^0, d^i(\mathbf{p}, x_i: f): f): f) \sim_i d^i(\mathbf{p}, x_i: f) \text{ for any } (\mathbf{p}, \mathbf{x}) \in A(f).$$

It holds by the condition (ii) of the admissibility of alternatives that $d^i(\mathbf{p}^0, x_i^0 + EV^i(\mathbf{p}, \mathbf{x}: g): g) \gg \mathbf{e}^0$. **(iii)** It holds by Lemma 2(ii) that $d^i(\mathbf{p}, x_i: f) \succsim_i d^i(\mathbf{q}, y_i: f) \Leftrightarrow d^i(\mathbf{p}^0, x_i^0 + EV^i(\mathbf{p}, \mathbf{x}: f): f) \succsim_i d^i(\mathbf{p}^0, x_i^0 + EV^i(\mathbf{q}, \mathbf{y}: f): f) \Leftrightarrow EV^i(\mathbf{p}, \mathbf{x}: f) \geq EV^i(\mathbf{q}, \mathbf{y}: f)$. **(iv)** It holds by (9) that $x_i/CR^i(\mathbf{p}, \mathbf{x}: f) = \mu^i(\mathbf{p}, d^i(\mathbf{p}^0, x_i^0: f): f)$ for any $(\mathbf{p}, \mathbf{x}) \in A(f)$, which implies that $d^i(\mathbf{p}, x_i/CR^i(\mathbf{p}, \mathbf{x}: f): f) = d^i(\mathbf{p}, \mu^i(\mathbf{p}, d^i(\mathbf{p}^0, x_i^0: f): f): f)$. Moreover, it holds by (8) that $d^i(\mathbf{p}, \mu^i(\mathbf{p}, d^i(\mathbf{p}^0, x_i^0: f): f): f) \sim_i d^i(\mathbf{p}^0, x_i^0: f)$. Hence we have that $d^i(\mathbf{p}^0, x_i^0: f) \sim_i d^i(\mathbf{p}, x_i/CR^i(\mathbf{p}, \mathbf{x}: f): f)$. It holds by the condition (i) of the admissibility of alternatives that $d^i(\mathbf{p}, x_i/CR^i(\mathbf{p}, \mathbf{x}: f): g) \gg \mathbf{e}^0$. **(v)** It holds by (10) and (8) that

$$d^i(\mathbf{p}^0, ER^i(\mathbf{p}, \mathbf{x}: f) \cdot x_i^0: f) = d^i(\mathbf{p}^0, \mu^i(\mathbf{p}^0, d^i(\mathbf{p}, x_i: f): f): f) \sim_i d^i(\mathbf{p}, x_i: f) \text{ for any } (\mathbf{p}, \mathbf{x}) \in A(f).$$

It holds by the condition (ii) of the admissibility of alternatives that $d^i(\mathbf{p}^0, ER^i(\mathbf{p}, \mathbf{x}: f) \cdot x_i^0: g) \gg \mathbf{e}^0$. **(vi)** It holds by Lemma 2(v) that $d^i(\mathbf{p}, x_i: f) \succsim_i d^i(\mathbf{q}, y_i: f) \Leftrightarrow d^i(\mathbf{p}^0, ER^i(\mathbf{p}, \mathbf{x}: f) \cdot x_i^0: f) \succsim_i d^i(\mathbf{p}^0, ER^i(\mathbf{q}, \mathbf{y}: f) \cdot x_i^0: f) \Leftrightarrow ER^i(\mathbf{p}, \mathbf{x}: f) \geq ER^i(\mathbf{q}, \mathbf{y}: f)$. □

Proof of Lemma 3: **(A)** Since it holds by Lemma 2(iii) that the indirect preference ordering of $\succsim_i$ on $A(f)$ is represented by $EV^i(\mathbf{p}, \mathbf{x}: f)$ for all $i \in N$, we have that

$$d^i(\mathbf{p}, x_i: f) \succ_i d^i(\mathbf{q}, y_i: f) \text{ for all } i \in N \Rightarrow EV^i(\mathbf{p}, \mathbf{x}: f) > EV^i(\mathbf{q}, \mathbf{y}: f) \text{ for all } i \in N$$

$$\Rightarrow (\mathbf{p}, \mathbf{x})\, W_S^{EV}(f)\, (\mathbf{q}, \mathbf{y}).$$

Since it holds by Lemma 2(vi) that the indirect preference ordering of $\succsim_i$ on $A(f)$ is represented by $ER^i(\mathbf{p}, \mathbf{x}: f)$ for all $i \in N$, we can prove that W^{ER} satisfies the Pareto axiom by almost the same manner as the proof Lemma 3(A) in case of W^{EV}.

(B) The two social ordering functions, W^{CV} and W^{CR} do not satisfy the Pareto axiom as shown by Example 3 in Appendix C. □

Proof of Lemma 4: **(A)** Suppose that $f = (\mathbf{p}^0, \mathbf{x}^0, \succsim) \in F$. Fix any $(\mathbf{p}, \mathbf{x}) \in A(f)$, and fix any permutation θ of N with $\theta \circ \mathbf{x} \neq \mathbf{x}$. Since it holds by (9) that $\Sigma_{i \in N} CV^i(\mathbf{p}, \mathbf{x}: f) = \Sigma_{i \in N} x_i - \Sigma_{i \in N} \mu^i(\mathbf{p}, d^i(\mathbf{p}^0, x_i^0 : f): f) = \Sigma_{i \in N} x_{\theta(i)} - \Sigma_{i \in N} \mu^i(\mathbf{p}, d^i(\mathbf{p}^0, x_i^0 : f): f) = \Sigma_{i \in N} CV^i(\mathbf{p}, \theta \circ \mathbf{x}: f)$, we have that $(\mathbf{p}, \mathbf{x})\ W_I^{CV}(f)\ (\mathbf{p}, \theta \circ \mathbf{x})$. This means that W^{CV} satisfies the symmetry axiom. Since it holds by (11) that $\Pi_{i \in N} CR^i(\mathbf{p}, \mathbf{x}: f) = \Pi_{i \in N} x_i / \Pi_{i \in N} \mu^i(\mathbf{p}, d^i(\mathbf{p}^0, x_i^0: f): f) = \Pi_{i \in N} x_{\theta(i)} / \Pi_{i \in N} \mu^i(\mathbf{p}, d^i(\mathbf{p}^0, x_i^0: f): f) = \Pi_{i \in N} CR^i(\mathbf{p}, \theta \circ \mathbf{x}: f)$, we have that $(\mathbf{p}, \mathbf{x})\ W_I^{CR}(f)\ (\mathbf{p}, \theta \circ \mathbf{x})$. This means that W^{CR} satisfies the symmetry axiom. **(B)** The social ordering functions W^{EV} and W^{ER} do not satisfy the symmetry axiom as shown by Example 4 in Appendix C. □

Proof of Lemma 5: For any profiles $f = (\mathbf{p}^0, \mathbf{x}^0, \succsim)$ and $f^* = (\mathbf{q}^0, \mathbf{y}^0, \succsim^*)$ in F, let $(\mathbf{p}^1, \mathbf{x}^1)$ and $(\mathbf{q}^1, \mathbf{y}^1)$ be alternative in A(f) and A(f*), respectively.

(A) Fix any $i \in N$ and suppose that

$$d^i(\mathbf{p}^1, x_i^1 + \delta_i : f) \succsim_i d^i(\mathbf{p}^0, x_i^0 : f)$$

$$\Leftrightarrow\ d^i(\mathbf{q}^1, y_i^1 + \delta_i: f^*) \succsim_i^* d^i(\mathbf{q}^0, y_i^0: f^*) \quad \text{for all } \delta_i \geq -\min(x_i^1, y_i^1). \tag{A1}$$

We may assume that $\min(x_i^1, y_i^1) = x_i^1$ without loss of generality. It holds by Lemma 2(i) that $d^i(\mathbf{p}^0, x_i^0: f) \sim_i d^i(\mathbf{p}, x_i - CV^i(\mathbf{p}^1, \mathbf{x}^1: f): f)$. Then it holds by (A1) that $d^i(\mathbf{q}^0, y_i^0: f^*) \sim_i^* d^i(\mathbf{q}^1, y_i^1 - CV^i(\mathbf{p}^1, \mathbf{x}^1: f): f^*)$, which implies $CV^i(\mathbf{p}^1, \mathbf{x}^1: f) = CV^i(\mathbf{q}^1, \mathbf{y}^1: f^*)$.

Conversely, suppose that $CV^i(\mathbf{p}^1, \mathbf{x}^1: f) = CV^i(\mathbf{q}^1, \mathbf{y}^1: f^*)$. Setting $\delta_i^* = -CV^i(\mathbf{p}^1, \mathbf{x}^1: f) = -CV^i(\mathbf{q}^1, \mathbf{y}^1: f^*)$, we have by Lemma 2(i) that

$$d^i(\mathbf{p}^1, x_i^1 + \delta_i^*: f) \sim_i d^i(\mathbf{p}^0, x_i^0: f) \tag{A2}$$

and

$$d^i(\mathbf{q}^1, y_i^1 + \delta_i^*: f^*) \sim_i^* d^i(\mathbf{q}^0, y_i^0: f^*). \tag{A3}$$

For a real number $\delta_i \geq -\min(x_i^1, y_i^1)$, suppose that $d^i(\mathbf{p}^1, x_i^1 + \delta_i: f) \succsim_i d^i(\mathbf{p}^0, x_i^0: f)$. Then it holds by (A2) that $\delta_i \geq \delta_i^*$, and it holds by this and (A3) that $d^i(\mathbf{q}^1, y_i^1 + \delta_i: f^*) \succsim_i^* d^i(\mathbf{q}^0, y_i^0: f^*)$. Similarly we can prove that $d^i(\mathbf{p}^1, x_i^1 + \delta_i: f) \succsim_i d^i(\mathbf{p}^0, x_i^0: f)$, whenever $d^i(\mathbf{q}^1, y_i^1 + \delta_i: f^*) \succsim_i^* d^i(\mathbf{q}^0, y_i^0: f^*)$.

(B) We can prove this assertion by almost the same manner as in the proof of the assertion (A).

(C) Fix any $i \in N$ and suppose that

$$d^i(\mathbf{p}^1, x_i^1 \cdot \delta_i : f) \precsim_i d^i(\mathbf{p}^0, x_i^0 : f) \Leftrightarrow d^i(\mathbf{q}^1, y_i^1 \cdot \delta_i : f^*) \precsim_i^* d^i(\mathbf{q}^0, y_i^0 : f^*) \quad \text{for all } \delta_i > 0. \qquad \text{(A4)}$$

It holds by Lemma 2(iv) that $d^i(\mathbf{p}^0, x_i^0 : f) \sim_i d^i(\mathbf{p}, x_i / CR^i(\mathbf{p}^1, \mathbf{x}^1 : f) : f)$. Then it holds by (A4) that $d^i(\mathbf{q}^0, y_i^0 : f^*) \sim_i^* d^i(\mathbf{q}^1, y_i^1 / CR^i(\mathbf{p}^1, \mathbf{x}^1 : f) : f^*)$, which implies $CR^i(\mathbf{p}^1, \mathbf{x}^1 : f) = CR^i(\mathbf{q}^1, \mathbf{y}^1 : f^*)$.

Conversely, suppose that $CR^i(\mathbf{p}^1, \mathbf{x}^1 : f) = CR^i(\mathbf{q}^1, \mathbf{y}^1 : f^*)$. Setting $1/CR^i(\mathbf{p}^1, \mathbf{x}^1 : f) = 1/ CR^i(\mathbf{q}^1, \mathbf{y}^1 : f^*) = \delta_i^*$, we have by Lemma 2(i) that

$$d^i(\mathbf{p}^1, x_i^1 \cdot \delta_i^* : f) \sim_i d^i(\mathbf{p}^0, x_i^0 : f) \qquad \text{(A5)}$$

and

$$d^i(\mathbf{q}^1, y_i^1 \cdot \delta_i^* : f^*) \sim_i^* d^i(\mathbf{q}^0, y_i^0 : f^*). \qquad \text{(A6)}$$

For a real number $\delta_i > 0$ suppose that $d^i(\mathbf{p}^1, x_i^1 \cdot \delta_i : f) \precsim_i d^i(\mathbf{p}^0, x_i^0 : f)$. Then it holds by (A5) that $\delta_i \geq \delta_i^*$, and it holds by this and (A6) that $d^i(\mathbf{q}^1, y_i^1 \cdot \delta_i : f^*) \precsim_i^* d^i(\mathbf{q}^0, y_i^0 : f^*)$. Similarly we can prove that $d^i(\mathbf{p}^1, x_i^1 \cdot \delta_i : f) \precsim_i d^i(\mathbf{p}^0, x_i^0 : f)$, whenever $d^i(\mathbf{q}^1, y_i^1 \cdot \delta_i : f^*) \precsim_i^* d^i(\mathbf{q}^0, y_i^0 : f^*)$.

(D) We can prove this assertion by almost the same manner as in the proof of the assertion (C). □

Proof Lemma 7: (A) Set $F^L(\mathbf{p}^0) = \{ (x_i^0, \precsim_i)_{i \in N} \in (\mathbb{R}_{++} \times \mathcal{M})^N : (\mathbf{p}^0, \mathbf{x}^0, \succsim) \in F^L \}$ and $C_i^L(\mathbf{p}^0) = \mathbf{Proj}_i F^L(\mathbf{p}^0)$ for all $i \in N$. Then it holds by the definition that $C_1^L(\mathbf{p}^0) = C_2^L(\mathbf{p}^0) = \cdots = C_n^L(\mathbf{p}^0)$, and that the set of all quasi-linear profiles F^L satisfies

$$F^L = \{ f = (\mathbf{p}^0, \mathbf{x}^0, \succsim) \in F : (x_i^0, \precsim_i)_{i \in N} \in C_1^L(\mathbf{p}^0) \times C_2^L(\mathbf{p}^0) \times \cdots \times C_n^L(\mathbf{p}^0) \}.$$

(B) (i ⇔ ii) This assertion is well-known. See Chipman and Moore (1980, Section III). **(i ⇒ iii)** It holds by the assertion (i ⇒ ii) above that for all $i \in N$ and all $(\mathbf{p}, \mathbf{x}), (\mathbf{q}, \mathbf{y}) \in A(f)$

$$d^i(\mathbf{p}, x_i : f) \sim_i d^i(\mathbf{q}, y_i : f) \Rightarrow d^i(\mathbf{p}, x_i + \delta : f) \sim_i d^i(\mathbf{q}, y_i + \delta : f) \quad \text{for all } \delta > 0. \qquad \text{(A7)}$$

Case 1($CV^i(\mathbf{p}, \mathbf{x} : f) > 0$): It holds by Lemma 2(i), (A7) and Lemma 2(ii) that

$$d^i(\mathbf{p}^0, x_i^0 + CV^i(\mathbf{p}, \mathbf{x} : f) : f) \sim_i d^i(\mathbf{p}, x_i : f) \sim_i d^i(\mathbf{p}^0, x_i^0 + EV^i(\mathbf{p}, \mathbf{x} : f) : f),$$

which implies that $CV^i(\mathbf{p}, \mathbf{x} : f) = EV^i(\mathbf{p}, \mathbf{x} : f)$.

Case 2($CV^i(\mathbf{p}, \mathbf{x} : f) < 0$): It holds by Lemma 2(ii), (A7) and Lemma 2(i) that

$$d^i(\mathbf{p}^0, x_i^0 + EV^i(\mathbf{p}, \mathbf{x} : f) - CV^i(\mathbf{p}, \mathbf{x} : f) : f) \sim_i d^i(\mathbf{p}, x_i - CV^i(\mathbf{p}, \mathbf{x} : f) : f) \sim_i d^i(\mathbf{p}^0, x_i^0 : f),$$

which implies that $CV^i(\mathbf{p}, \mathbf{x} : f) = EV^i(\mathbf{p}, \mathbf{x} : f)$.

Case 3($CV^i(\mathbf{p}, \mathbf{x} : f) = 0$): It holds by Lemma 2(i, ii) that $d^i(\mathbf{p}^0, x_i^0 : f) \sim_i d^i(\mathbf{p}, x_i : f) \sim_i d^i(\mathbf{p}^0, x_i^0 + EV^i(\mathbf{p}, \mathbf{x} : f) : f)$, which implies that $EV^i(\mathbf{p}, \mathbf{x} : f) = 0$.

(iii ⇒ vi) This assertion is a direct consequence of Lemma 2(iii). **(iv ⇒ ii)** This assertion is a direct consequence of Chipman and Moore (1980, Proposition P2), where the indirect utility function (Chipman and Moore, 1980, Equation (32)) coincides with our $CV^i(\mathbf{p}, \mathbf{x}: f)$ under the normalization of price vectors by $p_1 = 1$.

(C) Let F^* be an admissible domain such that $F^L \subset F^*$ and $F^* \neq F^L$. Fix any $f = (\mathbf{p}^0, \mathbf{x}^0, \succsim) \in F^*/F^L$. Since $f \notin F^L$, it holds by Lemma 7B(i $\Leftrightarrow$ iv) that there exist some $j \in N$ and some $(\mathbf{p}, \mathbf{x})$, $(\mathbf{q}, \mathbf{y}) \in A(f)$ such that

$$d^j(\mathbf{p}, x_j: f) \succsim_j d^j(\mathbf{q}, y_j: f) \text{ and } CV^i(\mathbf{p}, \mathbf{x}: f) < CV^i(\mathbf{q}, \mathbf{y}: f), \text{ or}$$

$$d^j(\mathbf{p}, x_j: f) \succ_j d^j(\mathbf{q}, y_j: f) \text{ and } CV^i(\mathbf{p}, \mathbf{x}: f) \leq CV^i(\mathbf{q}, \mathbf{y}: f).$$

Since d^j and CV^i are continuous, and since $(\mathbf{p}, \mathbf{x})$, $(\mathbf{q}, \mathbf{y}) \in A(f)$, we can assume without loss of generality that $d^j(\mathbf{p}, x_j: f) \succ_j d^j(\mathbf{q}, y_j: f)$ and $CV^i(\mathbf{p}, \mathbf{x}: f) < CV^i(\mathbf{q}, \mathbf{y}: f)$. Set $g = (\mathbf{p}^0, \mathbf{y}^0, \succsim^0) \in F$ by $y_i^0 = x_j^0$ and $\succsim_i^0 = \succsim_j$ for all $i \in N$, and set $\mathbf{x}^* = x_j \cdot \mathbf{e}$, $\mathbf{y}^* = y_j \cdot \mathbf{e}$, where $\mathbf{e} = (1, \ldots, 1)$. Since F^* is admissible, it holds that $g \in F^*$ and $d^i(\mathbf{p}, x_i^*: g) \succ_i d^i(\mathbf{q}, y_i^*: g)$ for all $i \in N$, and that $\sum_{i \in N} CV^i(\mathbf{p}, \mathbf{x}^*: g) < \sum_{i \in N} CV^i(\mathbf{q}, \mathbf{y}^*: g)$. This means that W^{CV} does not satisfy the Pareto axiom on *g*.

Next, we prove that there exists some profile $h \in F^*$ on which W^{EV} does not satisfy the symmetry axiom. Fix any $f = (\mathbf{p}^0, \mathbf{x}^0, \succsim) \in F^*/F^L$ again. Since $f = (\mathbf{p}^0, \mathbf{x}^0, \succsim) \notin F^L$, it holds by Lemma 7B(i $\Leftrightarrow$ iii) that $CV^j(\mathbf{p}, \mathbf{x}: f) \neq EV^j(\mathbf{p}, \mathbf{x}: f)$ for some $j \in N$ and some $(\mathbf{p}, \mathbf{x}) \in A(f)$. Setting $b = \mu^j(\mathbf{p}, d^j(\mathbf{p}^0, x_j^0: f): f) > 0$, $\lambda = CV^j(\mathbf{p}, \mathbf{x}: f)$ and $\delta = EV^j(\mathbf{p}, \mathbf{x}: f)$, it hold by $(\mathbf{p}, \mathbf{x}) \in A(f)$, (8) and Lemma 2(i, ii) that

$$\lambda \neq \delta, \mathbf{e}^0 \ll d^j(\mathbf{p}^0, x_j^0: f) \sim_j d^j(\mathbf{p}, b: f) \gg \mathbf{e}^0$$
$$\text{and } \mathbf{e}^0 \ll d^j(\mathbf{p}^0, x_j^0+\delta: f) \sim_j d^j(\mathbf{p}, x_j: f) = d^j(\mathbf{p}, b+\lambda: f) \gg \mathbf{e}^0. \quad \text{(A8)}$$

We need a claim:

Claim 1: There exists a quasi-linear profile $g^* = (\mathbf{p}^0, \mathbf{y}^0, \succsim^*) \in F^L$ such that $y_1^0 = y_2^0 = \cdots = y_n^0$, $\succsim_1^* = \succsim_2^* = \cdots = \succsim_n^*$ and $(\mathbf{p}, b \cdot \mathbf{e}) \in A(g^*)$.

Define a profile $h \in F$ by $h = (\mathbf{p}^0, \mathbf{z}, \succsim^0)$, where

$$z_i = x_j^0 \text{ if } i = j, \quad = y_i^0 \text{ otherwise}, \qquad \succsim_i^0 = \succsim_j \text{ if } i = j, \quad = \succsim_i^* \text{ otherwise.}$$

Since F^* is admissible and $F^L \subset F^*$, it holds by Claim 1 that $h \in F^*$. Moreover, it holds by (A8) and Claim 1 that $(\mathbf{p}, b \cdot \mathbf{e}) \in A(h)$. Setting $a = EV^i(\mathbf{p}, b: h)$ for all $i \neq j$, it holds by Lemma 2(ii) and

$g^* \in F^L$ that

$$\mathbf{e}^0 \ll d^i(\mathbf{p}^0, z_i+a\colon h) \sim_i^0 d^i(\mathbf{p}, b\colon h) \gg \mathbf{e}^0$$

$$\text{and } \mathbf{e}^0 \ll d^i(\mathbf{p}^0, z_i+a+\lambda\colon h) \sim_i^0 d^i(\mathbf{p}, b+\lambda\colon h) \gg \mathbf{e}^0 \quad \text{for all } i \neq j. \tag{A9}$$

Set $\mathbf{x}^*$ by

$$x_i^* = b+\lambda \quad \text{if } i=j$$
$$= b \quad \text{otherwise.}$$

Then it holds by (A8) and (A9) that $(\mathbf{p}, \mathbf{x}^*) \in A(h)$ and

$$EV^j(\mathbf{p}, \mathbf{x}^*\colon h) = \mu^j(\mathbf{p}^0, d^j(\mathbf{p}, b+\lambda\colon h)\colon h) - \mu^j(\mathbf{p}^0, d^j(\mathbf{p}^0, x_j^0\colon h)\colon h)$$
$$= \mu^j(\mathbf{p}^0, d^j(\mathbf{p}^0, x_j^0+\delta\colon h)\colon h) - x_j^0 = x_j^0+\delta - x_j^0 = \delta,$$
$$EV^i(\mathbf{p}, \mathbf{x}^*\colon h) = \mu^i(\mathbf{p}^0, d^i(\mathbf{p}, b\colon h)\colon h) - \mu^i(\mathbf{p}^0, d^i(\mathbf{p}^0, z_i\colon h)\colon h)$$
$$= \mu^i(\mathbf{p}^0, d^i(\mathbf{p}^0, z_i + a\colon h)\colon h) - z_i = z_i + a - z_i = a \quad \text{for all } i \neq j.$$

Fix any $i^* \neq j$, and define a permutation θ on N by $\theta(j) = i^*$, $\theta(i^*) = j$ and $\theta(i) = i$ for all $i \in N/\{i^*, j\}$. We have by (A8) and (A9) that $(\mathbf{p}, \theta\circ\mathbf{x}^*) \in A(h)$ and

$$EV^j(\mathbf{p}, \theta\circ\mathbf{x}^*\colon h) = \mu^j(\mathbf{p}^0, d^j(\mathbf{p}, b\colon h)\colon h) - \mu^j(\mathbf{p}^0, d^j(\mathbf{p}^0, x_j^0\colon h)\colon h) = \mu^j(\mathbf{p}^0, d^j(\mathbf{p}^0, x_j^0\colon h)\colon h) - x_j^0$$
$$= x_j^0 - x_j^0 = 0,$$
$$EV^{i^*}(\mathbf{p}, \theta\circ\mathbf{x}^*\colon h) = \mu^{i^*}(\mathbf{p}^0, d^{i^*}(\mathbf{p}, b+\lambda\colon h)\colon h) - z_i = \mu^{i^*}(\mathbf{p}^0, d^{i^*}(\mathbf{p}, z_i+a+\lambda\colon h)\colon h) - z_i$$
$$= z_i+a+\lambda - z_i = a+\lambda.$$

Thus it holds by (A8) that $\Sigma_{i\in N}EV^i(\mathbf{p}, \mathbf{x}^*\colon h) - \Sigma_{i\in N}EV^i(\mathbf{p}, \theta\circ\mathbf{x}^*\colon h) = \delta - \lambda \neq 0$, and that $(\mathbf{p}, \mathbf{x}^*)W_S^{EV}(h)(\mathbf{p}, \theta\circ\mathbf{x}^*)$ or $(\mathbf{p}, \theta\circ\mathbf{x}^*)W_S^{EV}(h)(\mathbf{p},\mathbf{x}^*)$, which means that W^{EV} does not satisfy the symmetry axiom on *h*.

(D) For two profiles $f = (\mathbf{p}^0, \mathbf{x}^0, \succsim)$, $g = (\mathbf{q}^0, \mathbf{y}^0, \succsim^*) \in F^L$, suppose that $\succsim = \succsim^*$. Fix any $i \in N$ and any $(\mathbf{p}, \mathbf{x}), (\mathbf{q}, \mathbf{y}) \in A(f) \cap A(g)$. It holds by (8) that

$$d^i(\mathbf{p}, \mu^i(\mathbf{p}, d^i(\mathbf{p}^0, x_i\colon f)\colon f)\colon f) \sim_i d^i(\mathbf{q}, \mu^i(\mathbf{q}, d^i(\mathbf{p}^0, x_i\colon f)\colon f)\colon f) \text{ and}$$
$$d^i(\mathbf{p}, \mu^i(\mathbf{p}, d^i(\mathbf{q}^0, y_i\colon g)\colon g)\colon g) \sim_i d^i(\mathbf{q}, \mu^i(\mathbf{q}, d^i(\mathbf{q}^0, y_i\colon g)\colon g)\colon g). \tag{A10}$$

Since $\succsim = \succsim^*$, we may assume without loss of generality that $d^i(\mathbf{p}, \mu^i(\mathbf{p}, d^i(\mathbf{p}^0, x_i\colon f)\colon f)\colon f) \succsim_i d^i(\mathbf{p}, \mu^i(\mathbf{p}, d^i(\mathbf{q}^0, y_i\colon g)\colon g)\colon g)$, which implies that $\mu^i(\mathbf{p}, d^i(\mathbf{p}^0, x_i\colon f)\colon f) \geq \mu^i(\mathbf{p}, d^i(\mathbf{q}^0, y_i\colon g)\colon g)$. It holds by Lemma 7B(i $\Leftrightarrow$ ii), $p_1 = q_1 = 1$ and (A10) that

$$\mu^i(\mathbf{p}, d^i(\mathbf{p}^0, x_i\colon f)\colon f) - \mu^i(\mathbf{p}, d^i(\mathbf{q}^0, y_i\colon g)\colon g) = \mu^i(\mathbf{q}, d^i(\mathbf{p}^0, x_i\colon f)\colon f) - \mu^i(\mathbf{q}, d^i(\mathbf{q}^0, y_i\colon g)\colon g),$$

which implies

$$CV^i(\mathbf{p}, \mathbf{x}: f) - CV^i(\mathbf{q}, \mathbf{y}: f) = [\, x^i - \mu^i(\mathbf{p}, d^i(\mathbf{p}^0, x_i^0 : f): f)\,] - [\, y^i - \mu^i(\mathbf{q}, d^i(\mathbf{p}^0, x_i^0 : f): f)\,]$$

$$= [\, x^i - \mu^i(\mathbf{p}, d^i(\mathbf{q}^0, y_i: g): g)\,] - [\, y^i - \mu^i(\mathbf{q}, d^i(\mathbf{q}^0, y_i: g): g)] = CV^i(\mathbf{p}, \mathbf{x}: g) - CV^i(\mathbf{q}, \mathbf{y}: g).$$

Hence we have that

$$(\mathbf{p}, \mathbf{x})\, W^{CV}(f)\, (\mathbf{q}, \mathbf{y}) \Leftrightarrow \Sigma_{i\in N} CV^i(\mathbf{p}, x_i: f) \geq \Sigma_{i\in N} CV^i(\mathbf{q}, y_i: f)$$

$$\Leftrightarrow \Sigma_{i\in N}[CV^i(\mathbf{p}, x_i: f) - CV^i(\mathbf{q}, y_i: f)] \geq 0 \Leftrightarrow \Sigma_{i\in N}[\, CV^i(\mathbf{p}, x_i: g) - CV^i(\mathbf{q}, y_i: g)] \geq 0$$

$$\Leftrightarrow \Sigma_{i\in N} CV^i(\mathbf{p}, x_i: g) \geq \Sigma_{i\in N} CV^i(\mathbf{q}, y_i: g) \Leftrightarrow (\mathbf{p}, \mathbf{x})\, W^{CV}(g)\, (\mathbf{q}, \mathbf{y}). \qquad \square$$

Proof Lemma 8: (A) Set $F^H(\mathbf{p}^0) = \{ (x_i^0, \precsim_i)_{i\in N} \in (\mathbb{R}_{++} \times \mathcal{M})^N : (\mathbf{p}^0, \mathbf{x}^0, \succsim) \in F^H \}$ and $C_i^H(\mathbf{p}^0) =$ **Proj** $_i\, F^H(\mathbf{p}^0)$ for all $i \in N$. Then it holds by the definition that $C_1^H(\mathbf{p}^0) = C_2^H(\mathbf{p}^0) = \cdots = C_n^H(\mathbf{p}^0)$, and that the set of all homothetic profiles F^H satisfies

$$F^H = \{ f = (\mathbf{p}^0, \mathbf{x}^0, \succsim) \in F : (x_i^0, \precsim_i)_{i\in N} \in C_1^H(\mathbf{p}^0) \times C_2^H(\mathbf{p}^0) \times \cdots \times C_n^H(\mathbf{p}^0) \}.$$

(B) (i $\Leftrightarrow$ ii) This assertion is well-known. See Chipman and Moore (1980, Page 939) and Chipman (1974, Theorem 2). **(i $\Rightarrow$ iii)** It holds by the assertion (i $\Rightarrow$ ii) above that for all $i \in N$ and all $(\mathbf{p}, \mathbf{x}), (\mathbf{q}, \mathbf{y}) \in A(f)$

$$d^i(\mathbf{p}, x_i: f) \sim_i d^i(\mathbf{q}, y_i: f) \Rightarrow d^i(\mathbf{p}, \delta x_i : f) \sim_i d^i(\mathbf{q}, \delta y_i : f) \text{ for all } \delta > 0.$$

For all $i \in N$ and all $(\mathbf{p}, \mathbf{x}) \in A(f)$, it holds by this and Lemma 2(iv) that $d^i(\mathbf{p}, x_i: f) \sim_i d^i(\mathbf{p}^0, x_i^0 \cdot CR^i(\mathbf{p}, \mathbf{x}: f) : f)$, and it holds Lemma 2(iv) that $d^i(\mathbf{p}, x_i: f) \sim_i d^i(\mathbf{p}^0, x_i^0 \cdot ER^i(\mathbf{p}, \mathbf{x}: f): f)$. Hence we have that $d^i(\mathbf{p}^0, x_i^0 \cdot CR^i(\mathbf{p}, \mathbf{x}: f): f) \sim_i d^i(\mathbf{p}^0, x_i^0 \cdot ER^i(\mathbf{p}, \mathbf{x}: f): f)$, which implies that $CR^i(\mathbf{p}, \mathbf{x}: f) = ER^i(\mathbf{p}, \mathbf{x}: f)$. **(iii $\Rightarrow$ iv)** This assertion is a direct consequence of Lemma 2(vi). **(iv $\Rightarrow$ ii)** This assertion is a direct consequence of Chipman and Moore (1980, Proposition H2), where the indirect utility function (Chipman and Moore, 1980, Equation (23)) coincides with our $CR^i(\mathbf{p}, \mathbf{x}: f)$.

(C) Let F^* be an admissible domain such that $F^H \subset F^*$ and $F^* \neq F^H$. Fix any $f = (\mathbf{p}^0, \mathbf{x}^0, \succsim) \in F^*/F^H$. Since $f \notin F^H$, it holds by Lemma 8B(i $\Leftrightarrow$ iv) that there exist some $j \in N$ and some $(\mathbf{p}, \mathbf{x})$, $(\mathbf{q}, \mathbf{y}) \in A(f)$ such that

$d^j(\mathbf{p}, x_j: f) \precsim_j d^j(\mathbf{q}, y_j: f)$ and $CR^i(\mathbf{p}, \mathbf{x}: f) < CR^i(\mathbf{q}, \mathbf{y}: f)$, or

$d^j(\mathbf{p}, x_j: f) \succ_j d^j(\mathbf{q}, y_j: f)$ and $CR^i(\mathbf{p}, \mathbf{x}: f) \leq CR^i(\mathbf{q}, \mathbf{y}: f)$.

Since d^j and CR^i are continuous, and since $(\mathbf{p}, \mathbf{x}), (\mathbf{q}, \mathbf{y}) \in A(f)$, we can assume without loss of generality that $d^j(\mathbf{p}, x_j: f) \succ_j d^j(\mathbf{q}, y_j: f)$ and $CR^i(\mathbf{p}, \mathbf{x}: f) < CR^i(\mathbf{q}, \mathbf{y}: f)$. Set $g = (\mathbf{p}^0, \mathbf{y}^0, \succsim^0) \in F$ by $y_i^0 = x_j^0$ and $\precsim_i^0 = \precsim_j$ for all $i \in N$, and set $\mathbf{x}^* = x_j \cdot \mathbf{e}$, $\mathbf{y}^* = y_j \cdot \mathbf{e}$. Since F^* is admissible, it holds that

$g \in F^*$ and $d^i(\mathbf{p}, x_i^*: g) \succ_i d^i(\mathbf{q}, y_i^*: g)$ for all $i \in N$, and that $\prod_{i\in N} CR^i(\mathbf{p}, \mathbf{x}^*: g) < \prod_{i\in N} CR^i(\mathbf{q}, \mathbf{y}^*: g)$. This means that W^{CR} does not satisfy the Pareto axiom on g.

Next, we prove that there exists some $h \in F^*$ on which W^{ER} does not satisfy the symmetry axiom. Fix any $f = (\mathbf{p}^0, \mathbf{x}^0, \succsim) \in F^*/F^H$ again. Since $f = (\mathbf{p}^0, \mathbf{x}^0, \succsim) \notin F^H$, it holds by Lemma 8B(i $\Leftrightarrow$ iii) that $CR^j(\mathbf{p}, \mathbf{x}: f) \neq ER^j(\mathbf{p}, \mathbf{x}: f)$ for some $j \in N$ and some $(\mathbf{p}, \mathbf{x}) \in A(f)$. Setting $b = \mu^j(\mathbf{p}, d^j(\mathbf{p}^0, x_j^0: f): f) > 0$, $\lambda = CR^j(\mathbf{p}, \mathbf{x}: f)$ and $\delta = ER^j(\mathbf{p}, \mathbf{x}: f)$, it holds by $(\mathbf{p}, \mathbf{x}) \in A(f)$, (8) and Lemma 2(iv, v) that

$$\lambda \neq \delta,\ \mathbb{e}^0 \ll d^j(\mathbf{p}^0, x_j^0: f) \sim_j d^j(\mathbf{p}, b: f) \gg \mathbb{e}^0$$

$$\text{and}\quad \mathbb{e}^0 \ll d^j(\mathbf{p}^0, x_j^0 \cdot \delta: f) \sim_j d^j(\mathbf{p}, x_j: f) = d^j(\mathbf{p}, b \cdot \lambda: f) \gg \mathbb{e}^0. \tag{A11}$$

Setting a utility function U on Y by $U(\mathbf{y}) = \prod_{k=1}^m y_k$, define a binary relation $\succsim^*$ on Y by $\mathbf{y}_1 \succsim^* \mathbf{y}_2 \Leftrightarrow U(\mathbf{y}_1) \geq U(\mathbf{y}_2)$ for any $\mathbf{y}_1, \mathbf{y}_2 \in Y$. Set $a = (\prod_{k=1}^m p_k^0 / p_k)^{1/m}$, and set a profile $h \in F$ by $h = (\mathbf{p}^0, \mathbf{z}, \succsim^0)$, where

$$z_i = x_j^0 \quad \text{if } i = j \qquad \succsim_i^0 = \succsim_j \quad \text{if } i = j$$
$$\quad = a \cdot b \quad \text{otherwise} \qquad\quad = \succsim_i^* \quad \text{otherwise.}$$

Then it holds that

$$\mathbb{e}^0 \ll d^i(\mathbf{p}, 1: h) \sim_i^0 d^i(\mathbf{p}^0, a: h) \gg \mathbb{e}^0,\ \mathbb{e}^0 \ll d^i(\mathbf{p}, b: h) \sim_i^0 d^i(\mathbf{p}^0, a \cdot b: h) \gg \mathbb{e}^0 \quad \text{and}$$

$$\mathbb{e}^0 \ll d^i(\mathbf{p}, b \cdot \lambda: h) \sim_i^0 d^i(\mathbf{p}^0, b \cdot \lambda \cdot a: h) \gg \mathbb{e}^0 \quad \text{for all } i \neq j. \tag{A12}$$

Since F^* is admissible and $F^L \subset F^*$, it holds by (A11) and (A12) that $h \in F^*$. Set $\mathbf{x}^*$ by

$$x_i^* = b \cdot \lambda \quad \text{if } i = j$$
$$\quad = b \quad \text{otherwise.}$$

Then it holds by (A11) and (A12) that $(\mathbf{p}, \mathbf{x}^*) \in A(h)$ and

$$ER^j(\mathbf{p}, \mathbf{x}^*: h) = \mu^j(\mathbf{p}^0, d^j(\mathbf{p}, b \cdot \lambda: h): h) / \mu^j(\mathbf{p}^0, d^j(\mathbf{p}^0, x_j^0: h): h) = \mu^j(\mathbf{p}^0, d^j(\mathbf{p}^0, x_j^0 \cdot \delta: h): h) / x_j^0$$
$$= (x_j^0 \cdot \delta) / x_j^0 = \delta,$$

$$ER^i(\mathbf{p}, \mathbf{x}^*: h) = \mu^i(\mathbf{p}^0, d^i(\mathbf{p}, b: h): h) / \mu^i(\mathbf{p}^0, d^i(\mathbf{p}^0, a \cdot b: h): h) = \mu^i(\mathbf{p}^0, d^i(\mathbf{p}^0, a \cdot b: h): h) / (a \cdot b)$$
$$= (a \cdot b) / (a \cdot b) = 1 \quad \text{for all } i \neq j.$$

Fix any $i^* \neq j$, and define a permutation θ on N by $\theta(j) = i^*$, $\theta(i^*) = j$ and $\theta(i) = i$ for all $i \in N/\{i^*, j\}$. Since U is homothetic, we have by (A11) and (A12) that $(\mathbf{p}, \theta \circ \mathbf{x}^*) \in A(h)$ and

$$ER^j(\mathbf{p}, \theta \circ \mathbf{x}^*: h) = \mu^j(\mathbf{p}^0, d^j(\mathbf{p}, b: h): h) / x_j^0 = \mu^j(\mathbf{p}^0, d^j(\mathbf{p}^0, x_j^0: h): h) / x_j^0 = x_j^0 / x_j^0 = 1,$$

$$ER^{i^*}(\mathbf{p}, \theta \circ \mathbf{x}^*: h) = \mu^{i^*}(\mathbf{p}^0, d^{i^*}(\mathbf{p}, b \cdot \lambda: h): h) / \mu^{i^*}(\mathbf{p}^0, d^{i^*}(\mathbf{p}^0, b \cdot a: h): h)$$
$$= \mu^{i^*}(\mathbf{p}^0, d^{i^*}(\mathbf{p}^0, b \cdot \lambda \cdot a: h): h) / (b \cdot a) = (b \cdot \lambda \cdot a) / (b \cdot a) = \lambda.$$

Thus it holds by (A11) that $\Pi_{i\in N}ER^i(\mathbf{p}, \mathbf{x}^*: h)/\Pi_{i\in N}ER^i(\mathbf{p}, \theta\circ\mathbf{x}^*: h) = \delta/\lambda \neq 1$, and that $(\mathbf{p}, \mathbf{x}^*)\, W_S^{ER}(h)\, (\mathbf{p}, \theta\circ\mathbf{x}^*)$ or $(\mathbf{p}, \theta\circ\mathbf{x}^*)\, W_S^{ER}(h)\, (\mathbf{p}, \mathbf{x}^*)$, which means that W^{ER} does not satisfy the symmetry axiom on h.

(D) For any profiles $f = (\mathbf{p}^0, \mathbf{x}^0, \succsim)$, $g = (\mathbf{q}^0, \mathbf{y}^0, \succsim^*) \in F^H$, suppose that $\succsim = \succsim^*$. It holds by $f, g \in F^H$ that $A(f) = A(g)$. Fix any $i \in N$ and any $(\mathbf{p}, \mathbf{x}), (\mathbf{q}, \mathbf{y}) \in A(f)$. It holds by (8) that

$$d^i(\mathbf{p}, \mu^i(\mathbf{p}, d^i(\mathbf{p}^0, x_i: f): f) : f) \sim_i d^i(\mathbf{q}, \mu^i(\mathbf{q}, d^i(\mathbf{p}^0, x_i: f): f) : f) \text{ and}$$

$$d^i(\mathbf{p}, \mu^i(\mathbf{p}, d^i(\mathbf{q}^0, y_i: g): g) : g) \sim_i d^i(\mathbf{q}, \mu^i(\mathbf{q}, d^i(\mathbf{q}^0, y_i: g): g) : g). \quad \text{(A13)}$$

It holds by Lemma 8B(i $\Leftrightarrow$ ii) and (A13) that

$$\mu^i(\mathbf{p}, d^i(\mathbf{p}^0, x_i: f): f) / \mu^i(\mathbf{p}, d^i(\mathbf{q}^0, y_i: g): g) = \mu^i(\mathbf{q}, d^i(\mathbf{p}^0, x_i: f): f) / \mu^i(\mathbf{q}, d^i(\mathbf{q}^0, y_i: g): g),$$

which implies that

$$CR^i(\mathbf{p}, \mathbf{x}: f) / CR^i(\mathbf{q}, \mathbf{y}: f) = [\, x^i/\mu^i(\mathbf{p}, d^i(\mathbf{p}^0, x_i^0: f): f)\,] / [\, y^i/\mu^i(\mathbf{q}, d^i(\mathbf{p}^0, x_i^0: f): f)\,]$$

$$= [\, x^i/\mu^i(\mathbf{p}, d^i(\mathbf{q}^0, y_i: g): g)\,]/[\, y^i/\mu^i(\mathbf{q}, d^i(\mathbf{q}^0, y_i: g): g)] = CR^i(\mathbf{p}, \mathbf{x}: g)/CR^i(\mathbf{q}, \mathbf{y}: g).$$

Hence we have that

$$(\mathbf{p}, \mathbf{x})\, W^{CR}(f)\, (\mathbf{q}, \mathbf{y}) \Leftrightarrow \Pi_{i\in N}CR^i(\mathbf{p}, x_i: f) \geq \Pi_{i\in N}CR^i(\mathbf{q}, y_i: f)$$

$$\Leftrightarrow \Pi_{i\in N}[\, CR^i(\mathbf{p}, x_i: f)/CV^i(\mathbf{q}, y_i: f)\,] \geq 1 \Leftrightarrow \Pi_{i\in N}[\, CR^i(\mathbf{p}, x_i: g)/CR^i(\mathbf{q}, y_i: g)\,] \geq 1$$

$$\Leftrightarrow \Pi_{i\in N} CR^i(\mathbf{p}, x_i: g) \geq \Pi_{i\in N} CR^i(\mathbf{q}, y_i: g) \Leftrightarrow (\mathbf{p}, \mathbf{x})\, W^{CR}(g)\, (\mathbf{q}, \mathbf{y}). \quad \square$$

Proof of Lemma 9: **(i)** Fix any $\omega, \mathbf{a}, \mathbf{b}, \mathbf{c} \in X$. Since $\mathbf{a}/\omega = (\mathbf{a}*\mathbf{c})/(\omega*\mathbf{c})$ and $\mathbf{b}/\omega = (\mathbf{b}*\mathbf{c})/(\omega*\mathbf{c})$, we have by the R-independence axiom that $\mathbf{a}W(\omega)\mathbf{b} \Leftrightarrow (\mathbf{a}*\mathbf{c})W(\omega*\mathbf{c})(\mathbf{b}*\mathbf{c})$. **(ii)** Define a binary relation H on $\mathbb{R}^n$ by

$$\mathbf{a}H\mathbf{b} \Leftrightarrow \mathrm{Exp}(\mathbf{a})\, W(\mathbf{e})\, \mathrm{Exp}(\mathbf{b}) \text{ for all } \mathbf{a}, \mathbf{b} \in \mathbb{R}^n. \quad \text{(A14)}$$

Since $W(\mathbf{e})$ is complete and transitive on X, it holds by (A14) that H is a complete and transitive binary relation on $\mathbb{R}^n$. The symmetric and asymmetric parts of H are denoted by H_I and H_S, respectively. Moreover, H has the following properties:

Claim 2: **(i)** If $a_i > b_i$ for all $i \in N$, then $\mathbf{a}H_S\mathbf{b}$. **(ii)** For any $\mathbf{a} \in \mathbb{R}^n$ and any $i, j \in N$ with $i \neq j$, let $\mathbf{b}$ be the vector in $\mathbb{R}^n$ defined by $b_i = b_j = (a_i + a_j)/2$ and $b_k = a_k$ for all $k \in N/\{i,j\}$. Then it holds that $\mathbf{a}H_I\mathbf{b}$. **(iii)** For any $\mathbf{a}^1, \mathbf{a}^2 \in \mathbb{R}^n$, and any $i, j \in N$ with $i \neq j$, if $a_i^1 + a_j^1 = a_i^2 + a_j^2$, and if $a_k^1 = a_k^2$ for all $k \in N/\{i, j\}$, then $\mathbf{a}^1H_I\mathbf{a}^2$. **(iv)** For any $\mathbf{a} \in \mathbb{R}^n$, it holds that $\mathbf{a}H_I\bar{\mathbf{a}}$, where $\bar{\mathbf{a}} = (\sum_{i=1}^n a^i)/n \in \mathbb{R}^n$. **(v)** $\sum_{i\in N}a_i > \sum_{i\in N}b_i \Rightarrow \mathbf{a}H_S\mathbf{b}$. **(vi)** $\sum_{i\in N}a_i = \sum_{i\in N}b_i \Rightarrow \mathbf{a}H_I\mathbf{b}$.

We have by Claim 2(v, vi) that $\sum_{i\in N} a_i \geq \sum_{i\in N} b_i \Rightarrow$ **aHb**. If $\sum_{i\in N} a_i < \sum_{i\in N} b_i$, it holds by Claim 2(v) that $\mathbf{b}H_S\mathbf{a}$. Taking the contraposition of this, it holds that $\mathbf{a}H\mathbf{b} \Rightarrow \sum_{i\in N} a_i \geq \sum_{i\in N} b_i$. Thus we have that $\sum_{i\in N} a_i \geq \sum_{i\in N} b_i \Leftrightarrow$ **aHb**, and we have by (A14) that $\sum_{i\in N} a_i \geq \sum_{i\in N} b_i \Leftrightarrow \text{Exp}(\mathbf{a})\, W(\mathbf{e})\, \text{Exp}(\mathbf{b})$. □

Proof of Lemma 10: Fix any profile $f = (\mathbf{p}^0, \mathbf{x}^0, \succsim^0) \in F^L$, and fix any $(\mathbf{p}, \mathbf{x}), (\mathbf{q}, \mathbf{y}) \in A(f)$. Define **a** and **b** in $\mathbb{R}^n$ by

$$a_i = CV^i(\mathbf{p}, \mathbf{x}: f) \text{ and } b_i = CV^i(\mathbf{q}, \mathbf{y}: f) \text{ for all } i \in N. \tag{A15}$$

Define a profile $g = (\mathbf{p}^0, \omega, \succsim)$, where

$$\omega_i = 2 \cdot \max[\, \max_{i\in N} |a_i|, \max_{i\in N} |b_i|, \max_{i\in N} x_1^0 \,] \text{ and } \succsim_i = \succsim_1^0 \text{ for all } i \in N. \tag{A16}$$

Since $\omega_i > x_1^0$ for all $i \in N$, and since all $\succsim_i = \succsim_1^0$ are quasi-linear with respect to the #1 good, it holds by $f \in F^L$ and Lemma 7B(i ⇒ ii) that $d^i(\mathbf{p}^0, \omega_i: g) \gg \mathbf{e}^0$ for all $i \in N$ and $g \in F^L$. Similarly we can prove that $(\mathbf{p}^0, \omega+\mathbf{a}), (\mathbf{p}^0, \omega+\mathbf{b}) \in A(g)$. Hence we have by (A15) and (A16) that

$$CV^i(\mathbf{p}^0, \omega+\mathbf{a}: g) = CV^i(\mathbf{p}, \mathbf{x}: f) \text{ and } CV^i(\mathbf{p}^0, \omega+\mathbf{b}: g) = CV^i(\mathbf{q}, \mathbf{y}: f) \text{ for all } i \in N.$$

Since W satisfies the CV-independence axiom, it holds by this and Lemma 5(A) that

$$(\mathbf{p}^0, \omega+\mathbf{a})\, W(g)\, (\mathbf{p}^0, \omega+\mathbf{b}) \Leftrightarrow (\mathbf{p}, \mathbf{x})\, W(f)\, (\mathbf{q}, \mathbf{y}). \tag{A17}$$

Since $g \in F^L$ and $(\mathbf{p}^0, \omega+\mathbf{a}), (\mathbf{p}^0, \omega+\mathbf{b}) \in A(f)$, it holds that

$$(\mathbf{p}^0, \eta+\omega, \succsim) \in F^L \text{ and } (\mathbf{p}^0, \mathbf{z}+\omega) \in A(\mathbf{p}^0, \eta+\omega, \succsim) \text{ for all } \eta, \mathbf{z} \in X.$$

Hence we can define a social ordering function V on X by

$$\mathbf{z}^1 V(\eta)\, \mathbf{z}^2 \Leftrightarrow (\mathbf{p}^0, \mathbf{z}^1+\omega)\, W(\mathbf{p}^0, \eta+\omega, \succsim)\, (\mathbf{p}^0, \mathbf{z}^2+\omega) \text{ for all } \eta, \mathbf{z}^1, \mathbf{z}^2 \in X. \tag{A18}$$

Then the social ordering function V on X satisfies the Pareto, symmetry and A-independence axioms in Theorem 1(A), as shown by the following arguments: (Pareto): For any $\mathbf{x}, \mathbf{y}, \eta \in X$, suppose that $x_i > y_i$ for all $i \in N$. Since W satisfies the Pareto axiom, it holds that $(\mathbf{p}^0, \mathbf{x}+\omega)\, W_S(\mathbf{p}^0, \eta+\omega, \succsim)\, (\mathbf{p}^0, \mathbf{y}+\omega)$, which implies $\mathbf{x}V(\eta)\,\mathbf{y}$. (Symmetry): Since W satisfies the symmetry axiom, it holds by (A16) and (A17) that

$$(\mathbf{p}^0, \mathbf{x}+\omega)\, W_I(\mathbf{p}^0, \eta+\omega, \succsim)\, (\mathbf{p}^0, \theta\circ\mathbf{x}+\omega) \text{ and } \mathbf{x}V(\eta)\,\theta\circ\mathbf{x}$$

for any $\eta, \mathbf{x} \in X$ and any permutation θ of N. (A-Independence): Suppose that $\mathbf{x}^1 - \eta^1 = \mathbf{y}^1 - \eta^2$ and $\mathbf{x}^2 - \eta^1 = \mathbf{y}^2 - \eta^2$. It holds by (9) that

$$CV^i(\mathbf{p}^0, \mathbf{x}^1+\omega : \mathbf{p}^0, \eta^1+\omega, \succsim) = x_i^1 - \eta_i^1 = y_i^1 - \eta_i^2 = CV^i(\mathbf{p}^0, \mathbf{y}^1+\omega : \mathbf{p}^0, \eta^2+\omega, \succsim) \text{ and}$$

$$CV^i(\mathbf{p}^0, \mathbf{x}^2+\omega : \mathbf{p}^0, \eta^1+\omega, \succsim) = x_i^2 - \eta_i^1 = y_i^2 - \eta_i^2 = CV^i(\mathbf{p}^0, \mathbf{y}^2+\omega : \mathbf{p}^0, \eta^2+\omega, \succsim) \text{ for all } i \in N.$$

Since W satisfies the CV-independence axiom, it holds by Lemma 5(A) that

$$(\mathbf{p}^0, \mathbf{x}^1+\omega)\, W(\mathbf{p}^0, \eta^1+\omega, \succsim)\, (\mathbf{p}^0, \mathbf{x}^2+\omega) \Leftrightarrow (\mathbf{p}^0, \mathbf{y}^1+\omega)\, W(\mathbf{p}^0, \eta^2+\omega, \succsim)\, (\mathbf{p}^0, \mathbf{y}^2+\omega),$$

which implies that $\mathbf{x}^1\, V(\eta^1)\, \mathbf{x}^2 \Leftrightarrow \mathbf{y}^1\, V(\eta^2)\, \mathbf{y}^2$. Thus V satisfies all the axioms in Theorem 1(A), and it holds by Theorem 1(A) that V coincides with the arithmetic mean social ordering function W^A. Hence we have by (A18) that

$$\begin{aligned}\textstyle\sum_{i\in N} z_i^1 \geq \sum_{i\in N} z_i^2 &\Leftrightarrow \mathbf{z}^1\, V(\eta)\, \mathbf{z}^2 \\ &\Leftrightarrow (\mathbf{p}^0, \mathbf{z}^1+\omega)\, W(\mathbf{p}^0, \eta+\omega, \succsim)\, (\mathbf{p}^0, \mathbf{z}^2+\omega) \quad \text{for all } \eta, \mathbf{z}^1, \mathbf{z}^2 \in X. \qquad \text{(A19)}\end{aligned}$$

Setting $\eta = \omega$, $\mathbf{z}^1 = \omega+\mathbf{a}$, $\mathbf{z}^2 = \omega+\mathbf{b}$ in (A19), we have that $\sum_{i\in N} a_i \geq \sum_{i\in N} b_i \Leftrightarrow (\mathbf{p}^0, \omega+\mathbf{a}+\omega)\, W(\mathbf{p}^0, \omega+\omega, \succsim)\, (\mathbf{p}^0, \omega+\mathbf{b}+\omega)$. Since $CV^i(\mathbf{p}^0, \omega+\mathbf{a}+\omega: \mathbf{p}^0, \omega+\omega, \succsim) = a_i = CV^i(\mathbf{p}^0, \omega+\mathbf{a}: \mathbf{p}^0, \omega, \succsim)$ and $CV^i(\mathbf{p}^0, \omega+\mathbf{b}+\omega: \mathbf{p}^0, \omega+\omega, \succsim) = b_i = CV^i(\mathbf{p}^0, \omega+\mathbf{b}: \mathbf{p}^0, \omega, \succsim)$ for all $i \in N$, we have by this and Lemma 5(A) that

$$\textstyle\sum_{i\in N} a_i \geq \sum_{i\in N} b_i \Leftrightarrow (\mathbf{p}^0, \omega+\mathbf{a}+\omega)\, W(\mathbf{p}^0, \omega+\omega, \succsim)\, (\mathbf{p}^0, \omega+\mathbf{b}+\omega) \Leftrightarrow (\mathbf{p}^0, \omega+\mathbf{a})\, W(\mathbf{p}^0, \omega, \succsim)\, (\mathbf{p}^0, \omega+\mathbf{b}).$$

Moreover, we have by this, (A15) and (A17) that

$$\textstyle\sum_{i\in N} CV^i(\mathbf{p}, \mathbf{x}: f) \geq \sum_{i\in N} CV^i(\mathbf{q}, \mathbf{y}: f) \Leftrightarrow (\mathbf{p}^0, \omega+\mathbf{a})\, W(\mathbf{p}^0, \omega, \succsim)\, (\mathbf{p}^0, \omega+\mathbf{b}) \Leftrightarrow (\mathbf{p}, \mathbf{x}) W(f) (\mathbf{q}, \mathbf{y}). \quad \square$$

Proof of Lemma 11: Fix any profile $f = (\mathbf{p}^0, \mathbf{x}^0, \succsim^0) \in F^H$, and fix any $(\mathbf{p}, \mathbf{x}), (\mathbf{q}, \mathbf{y}) \in A(f)$. Define $\mathbf{a}$ and $\mathbf{b}$ in $\mathbb{R}^n$ by

$$a_i = CR^i(\mathbf{p}, \mathbf{x}: f) \text{ and } b_i = CR^i(\mathbf{q}, \mathbf{y}: f) \text{ for all } i \in N. \qquad \text{(A20)}$$

Define a profile $g = (\mathbf{p}^0, \omega, \succsim)$ by

$$\omega_i = x_1^0 \text{ and } \succsim_i = \succsim_1^0 \text{ for all } i \in N. \qquad \text{(A21)}$$

It holds by $f \in F^H$ that $d^i(\mathbf{p}^0, \omega_i: g) \gg \mathbf{e}^0$ for all $i \in N$ and that $g \in F^H$. Similarly we can prove $A(f) \subset A(g)$ and $(\mathbf{p}^0, \omega*\mathbf{a}), (\mathbf{p}^0, \omega*\mathbf{b}) \in A(g)$. Hence we have by (11) and (15) that

$$CR^i(\mathbf{p}^0, \omega*\mathbf{a}: g) = CR^i(\mathbf{p}, \mathbf{x}: f) \text{ and } CR^i(\mathbf{p}^0, \omega*\mathbf{b}: g) = CR^i(\mathbf{q}, \mathbf{y}: f) \quad \text{for all } i \in N.$$

Since W satisfies the CR-independence axiom, it holds by Lemma 5(C) that

$$(\mathbf{p}^0, \omega*\mathbf{a})\, W(g)\, (\mathbf{p}^0, \omega*\mathbf{b}) \Leftrightarrow (\mathbf{p}, \mathbf{x})\, W(f)\, (\mathbf{q}, \mathbf{y}). \qquad \text{(A22)}$$

Since $(\mathbf{p}^0, \mathbf{z}) \in A(\mathbf{p}^0, \eta, \succsim)$ for all η, $\mathbf{z} \in X$ by $f \in F^H$, we can define a social ordering function V on X by

$$\mathbf{z}^1 V(\eta)\, \mathbf{z}^2 \Leftrightarrow (\mathbf{p}^0, \mathbf{z}^1)\, W(\mathbf{p}^0, \eta, \succsim)\, (\mathbf{p}^0, \mathbf{z}^2) \quad \text{for all } \eta, \mathbf{z}^1, \mathbf{z}^2 \in X. \qquad \text{(A23)}$$

The social ordering function V on X satisfies the Pareto, symmetry and R-independence axioms in Theorem 3, as shown by the following arguments: (Pareto) For any $\mathbf{x}, \mathbf{y}, \eta \in X$, suppose that $x_i > y_i$ for all $i \in N$. Since W satisfies the Pareto axiom, it holds that $(\mathbf{p}^0, \mathbf{x})\, W_S(\mathbf{p}^0, \eta, \succsim)\, (\mathbf{p}^0, \mathbf{y})$, which implies $\mathbf{x} V(\eta)\, \mathbf{y}$. (Symmetry): Let η^* be a distribution in X. Since W satisfies the symmetry axiom, it holds by (A21) that $(\mathbf{p}^0, \mathbf{x})\, W_I(\mathbf{p}^0, \eta^*, \succsim)\, (\mathbf{p}^0, \theta\circ\mathbf{x})$ and $\mathbf{x} V(\eta^*)\, \theta\circ\mathbf{x}$ for any $\mathbf{x} \in X$ and any permutation θ of N. (R-Independence): Suppose that $\mathbf{x}^1/\eta^1 = \mathbf{y}^1/\eta^2$ and $\mathbf{x}^2/\eta^1 = \mathbf{y}^2/\eta^2$. It holds by (11) that

$$CV^i(\mathbf{p}^0, \mathbf{x}^1\!: \mathbf{p}^0, \eta^1, \succsim) = x_i^1/\eta_i^1 = y_i^1/\eta_i^2 = CV^i(\mathbf{p}^0, \mathbf{y}^1\!: \mathbf{p}^0, \eta^2, \succsim) \quad \text{and}$$

$$CV^i(\mathbf{p}^0, \mathbf{x}^2\!: \mathbf{p}^0, \eta^1, \succsim) = x_i^2/\eta_i^1 = y_i^2/\eta_i^2 = CV^i(\mathbf{p}^0, \mathbf{y}^2\!: \mathbf{p}^0, \eta^2, \succsim) \quad \text{for all } i \in N.$$

Since W satisfies the CR-independence axiom, it holds by Lemma 5(C) that

$$(\mathbf{p}^0, \mathbf{x}^1)\, W(\mathbf{p}^0, \eta^1, \succsim^0)\, (\mathbf{p}^0, \mathbf{x}^2) \Leftrightarrow (\mathbf{p}^0, \mathbf{y}^1)\, W(\mathbf{p}^0, \eta^2, \succsim^0)\, (\mathbf{p}^0, \mathbf{y}^2)$$

$$\text{and} \quad \mathbf{x}^1 V(\eta^1)\, \mathbf{x}^2 \Leftrightarrow \mathbf{y}^1 V(\eta^2)\, \mathbf{y}^2.$$

Thus V satisfies all the axioms in Theorem 1(B), and it holds by Theorem 1(B) that V coincides with the geometric mean social ordering function W^G. Hence we have by (A23) that

$$\Pi_{i\in N} z_i^1 \geq \Pi_{i\in N} z_i^2 \Leftrightarrow \mathbf{z}^1 V(\eta)\, \mathbf{z}^2$$

$$\Leftrightarrow (\mathbf{p}^0, \mathbf{z}^1)\, W(\mathbf{p}^0, \eta, \succsim)\, (\mathbf{p}^0, \mathbf{z}^2) \quad \text{for all } \eta, \mathbf{z}^1, \mathbf{z}^2 \in X. \tag{A24}$$

Setting $\eta = \omega$, $\mathbf{z}^1 = \omega*\mathbf{a}$ and $\mathbf{z}^2 = \omega*\mathbf{b}$ in (A24), we have by (A21) that

$$\Pi_{i\in N} a_i \geq \Pi_{\in N} b_i \Leftrightarrow \Pi_{i\in N} x_1^0\cdot a_i \geq \Pi_{\in N} x_1^0\cdot b_i \Leftrightarrow (\mathbf{p}^0, \omega*\mathbf{a})\, W(\mathbf{p}^0, \omega, \succsim)\, (\mathbf{p}^0, \omega*\mathbf{b}),$$

and it holds by this, (A20) and (A22) that

$$\Pi_{i\in N} CR^i(\mathbf{p}, \mathbf{x}\!: f) \geq \Pi_{i\in N} CR^i(\mathbf{q}, \mathbf{y}\!: f) \Leftrightarrow (\mathbf{p}^0, \omega*\mathbf{a})\, W(g)\, (\mathbf{p}^0, \omega*\mathbf{b}) \Leftrightarrow (\mathbf{p}, \mathbf{x})\, W(f)\, (\mathbf{q}, \mathbf{y}). \quad \square$$

Appendix B

Proof of Claim 1: Set $U(\mathbf{x}) = \Pi_{i=1}^m x_i$. Define a binary relation $\succsim$ on $\mathbb{R}^m_{++}$ by

$$\mathbf{x} \succsim \mathbf{y} \Leftrightarrow \Pi_{i=1}^m x_i \geq \Pi_{i=1}^m y_i \quad \text{for any } \mathbf{x}, \mathbf{y} \in \mathbb{R}^m_{++}.$$

Denote the demand function of $\succsim$ by $D(\mathbf{p}, x : \succsim)$. Since $\succsim$ is homothetic, it holds that

$$D(\mathbf{p}, b\!: \succsim) \gg \mathbf{e}^0 \text{ and } D(\mathbf{p}, b\!: \succsim) \sim D(\mathbf{p}^0, z^*\!: \succsim) \gg \mathbf{e}^0 \text{ for some } z^* > 0.$$

For each $\varepsilon > 0$, set $U^\varepsilon(\mathbf{x}) = \Pi_{i=1}^m (x_i+\varepsilon)$, and define a binary relation $\succsim^\varepsilon$ on $\{\mathbf{z} \in \mathbb{R}^m\!: \mathbf{z} \gg (-\varepsilon)\cdot\mathbf{e}\}$ by

$$\mathbf{x} \succsim^\varepsilon \mathbf{y} \Leftrightarrow \Pi_{i=1}^m (x_i + \varepsilon) \geq \Pi_{i=1}^m (y_i + \varepsilon) \quad \text{for any } \mathbf{x}, \mathbf{y} \in \{\mathbf{z} \in \mathbb{R}^m\!: \mathbf{z} \gg (-\varepsilon)\cdot\mathbf{e}\}$$

Denote the demand function of $\succsim^\varepsilon$ by $D(\mathbf{p}, x\!: \succsim^\varepsilon)$. Since $U^\varepsilon(\mathbf{x})$ converges uniformly to $U(\mathbf{x})$ on any compacta in $\mathbb{R}^m_{++}$ as $\varepsilon \to 0$, there exists some $\varepsilon > 0$ such that

$$D(\mathbf{p}, b: \succsim^\varepsilon) \gg \mathbf{e}^0 \text{ and } D(\mathbf{p}, b: \succsim^\varepsilon) \sim^\varepsilon D(\mathbf{p}^0, z^\varepsilon: \succsim^\varepsilon) \gg \mathbf{e}^0 \text{ for some } z^\varepsilon > 0.$$

Fix an $\varepsilon > 0$ and $z^\varepsilon > 0$ satisfying the above condition. Setting the real number $A^* \equiv U^\varepsilon(D(\mathbf{p}, b: \succsim^\varepsilon))$, define a real-valued function $V(\mathbf{x})$ on $\mathbb{R}^m_+$ by $V(\mathbf{x}) = c_\mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^m_+$, where $c_\mathbf{x}$ is the real number determined by $U^\varepsilon(x_1 - c_\mathbf{x}, x_2, \cdots, x_m) = A^*$. Moreover, define a binary relation $\succsim^*$ on $\mathbb{R}^m_+$ by $\mathbf{x} \succsim^* \mathbf{y} \Leftrightarrow V(\mathbf{x}) \geq V(\mathbf{y})$ for any $\mathbf{x}, \mathbf{y} \in \mathbb{R}^m_+$, and define a profile $g^* = (\mathbf{p}^0, \mathbf{y}^0, \boldsymbol{\succsim}^*) \in F$ by $y^0_1 = y^0_2 = \cdots = y^0_n = z^\varepsilon$ and $\succsim^*_1 = \succsim^*_2 = \cdots = \succsim^*_n = \succsim^*$. Then it holds that $g^* = (\mathbf{p}^0, \mathbf{y}^0, \boldsymbol{\succsim}^*) \in F^L$, $d^i(\mathbf{p}, b: g^*) \gg \mathbf{e}^0$ and $d^i(\mathbf{p}, b: g^*) \sim^*_i d^i(\mathbf{p}^0, y^0_i: g^*) \gg \mathbf{e}^0$ for all $i \in N$, which implies that $(\mathbf{p}, b \cdot \mathbf{e}) \in A(g^*)$. □

Proof of Claim 2: **(i)** Suppose that $a_i > b_i$ for all $i \in N$. Since $e^{a_i} > e^{b_i}$ for all $i \in N$, it holds by the Pareto axiom that $\text{Exp}(\mathbf{a})\, W(\mathbf{e})\, \text{Exp}(\mathbf{b})$. Hence we have by (A14) that $\mathbf{a} H_S \mathbf{b}$. **(ii)** For notational simplicity, we assume $i = 1$ and $j = 2$. Fix any $\mathbf{a} \in \mathbb{R}^n$, and let $\mathbf{b}$ be the vector in $\mathbb{R}^n$ defined by

$$b_1 = b_2 = (1/2)\, a_1 + (1/2)\, a_2 \text{ and } b_k = a_k \text{ for all } k \in \{3, 4, \cdots, n\}.$$

Then we will prove that $\mathbf{a} H_I \mathbf{b}$ in the followings. Define two alternatives $\mathbf{z}, \mathbf{z}^* \in X$ by

$$\mathbf{z} = \text{Exp}((1/4)\mathbf{a} - (1/4)\mathbf{b}) \text{ and } \mathbf{z}^* = \text{Exp}(-(1/4)\mathbf{a} + (1/4)\mathbf{b})$$

and define two alternatives $\mathbf{x}, \mathbf{y} \in X$ by

$$\mathbf{x} = \text{Exp}((3/4)\mathbf{a} + (1/4)\mathbf{b}) \text{ and } \mathbf{y} = \text{Exp}(-(1/4)\mathbf{a} + (5/4)\mathbf{b}).$$

Then it holds that

$$\mathbf{x}/\mathbf{z} = \text{Exp}((1/2)\mathbf{a} + (1/2)\mathbf{b}) \text{ and } \mathbf{y}/\mathbf{z} = \text{Exp}(-(1/2)\mathbf{a} + (3/2)\mathbf{b}), \tag{B1}$$

and that

$$\mathbf{x}/\mathbf{z}^* = \text{Exp}(\mathbf{a}) \text{ and } \mathbf{y}/\mathbf{z}^* = \text{Exp}(\mathbf{b}). \tag{B2}$$

Define $\mathbf{c} \in \mathbb{R}^n$ by

$$\mathbf{c} = \text{Log}(\mathbf{x}/\mathbf{z}) \tag{B3}$$

Then it holds that

$$\mathbf{c} = (1/2)\mathbf{a} + (1/2)\mathbf{b}. \tag{B4}$$

In fact, $c_1 = (1/2)\, a_1 + (1/2) b_1 = (3/4) a_1 + (1/4) a_2$, $c_2 = (1/2)\, a_2 + (1/2) b_2 = (1/4) a_1 + (3/4) a_2$ and $c_k = a_k$ for all $k \in \{3, 4, \cdots, n\}$. Let θ be a permutation defined by $\theta(1) = 2$, $\theta(2) = 1$, $\theta(i) = i$ for $i \in \{3, 4, \cdots, n\}$. Then it holds by (B1) and (B4) that

$$\theta \circ \mathbf{c} = -(1/2)\mathbf{a} + (3/2)\mathbf{b} = \text{Log}(\mathbf{y}/\mathbf{z})\,. \tag{B5}$$

In fact, $(\theta\circ\mathbf{c})_1 = c_2 = (1/4)a_1+(3/4)a_2 = -(1/2)a_1+(3/2)[a_1+a_2]/2 = -(1/2)a_1+(3/2)b_1$, $(\theta\circ\mathbf{c})_2 = c_1 = (3/4)a_1+(1/4)a_2 = -(1/2)a_2+(3/2)[a_1+a_2]/2 = -(1/2)a_2+(3/2)b_2$ and $(\theta\circ\mathbf{c})_k = c_k = a_k = b_k$ for all $k \in \{3, 4, \cdots, n\}$. Hence it holds (B3), (B5) that $\theta\circ \mathrm{Log}(\mathbf{x}/\mathbf{z}) = \mathrm{Log}(\mathbf{y}/\mathbf{z})$, which implies that $\theta\circ(\mathbf{x}/\mathbf{z}) = (\mathbf{y}/\mathbf{z})$. Since W satisfies the symmetry axiom, we have that $(\mathbf{x}/\mathbf{z})W_I(\mathbf{z}^*/\mathbf{z})(\mathbf{y}/\mathbf{z})$. Hence we have by Lemma 9(i) that $\mathbf{x}W_I(\mathbf{z}^*)\,\mathbf{y}$ and $(\mathbf{x}/\mathbf{z}^*)\,W_I(\mathbf{e})\,(\mathbf{y}/\mathbf{z}^*)$. It holds by (B2) that $\mathrm{Exp}(\mathbf{a})\,W(\mathbf{e})\,\mathrm{Exp}(\mathbf{b})$. Thus we have by (A14) that $\mathbf{a}H_I\mathbf{b}$. **(iii)** Define a vector $\mathbf{b} \in \mathbb{R}^n$ by $b_i = b_j = (a_i^1 + a_j^1)/2$ and $b_k = a_k^1$ for all $k \in N/\{i, j\}$. Then it holds by Claim 2(ii) that $\mathbf{a}^1H_I\mathbf{b}$. Since $b_i = b_j = (a_i^1 + a_j^1)/2 = (a_i^2 + a_j^2)/2$ and $b_k = a_k^1 = a_k^2$ for all $k \in N/\{i, j\}$, it holds by Claim 2(ii) that $\mathbf{a}^2 H_I\mathbf{b}$. Thus we have by $\mathbf{a}^1H_I\mathbf{b}$ and $\mathbf{a}^2 H_I\mathbf{b}$ that $\mathbf{a}^1H_I\mathbf{a}^2$. **(iv)** Fix any $\mathbf{a} \in \mathbb{R}^n$. Setting $\mu = (\Sigma_{i\in N} a_i)/n$, define a finite sequence of vectors by

$$\mathbf{a}^1 = (\mu, a_1 + a_2 - \mu, a_3, \cdots, a_n)$$
$$\mathbf{a}^2 = (\mu, \mu, a_1+a_2+a_3 - 2\mu, a_4, \cdots, a_n)$$
$$\cdots$$
$$\mathbf{a}^k = (\mu, \mu, \cdots, \mu, a_1+a_2+a_3+a_{k+1} - k\mu, a_{k+2}, \cdots, a_n)$$
$$\cdots$$
$$\mathbf{a}^{n-1} = (\mu, \mu, \cdots, \mu, a_1+a_2+a_3+a_n-(n-1)\mu) = (\mu, \mu, \cdots, \mu) = \bar{\mathbf{a}}.$$

Then we have by Claim 2(iii) that $\mathbf{a}H_I\mathbf{a}^1H_I\mathbf{a}^2H_I\cdots H_I\mathbf{a}^{n-2}H_I\bar{\mathbf{a}}$, which implies $\mathbf{a}H_I\bar{\mathbf{a}}$. **(v)** Fix any $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$. Set $\bar{\mathbf{a}} = [(\Sigma_{i\in N}a_i)/n]\cdot\mathbf{e} \in \mathbb{R}^n$ and $\bar{\mathbf{b}} = [(\Sigma_{i\in N}b_i)/n]\cdot\mathbf{e} \in \mathbb{R}^n$. Then we have by Claim 2(iv) that $\bar{\mathbf{a}}H_I\mathbf{a}$ and $\bar{\mathbf{b}}H_I\mathbf{b}$. Since $\Sigma_{i\in N}a_i > \Sigma_{i\in N}b_i$, we have by Claim 2(i) that $\bar{\mathbf{a}}H_S\bar{\mathbf{b}}$. Thus we have that $\mathbf{a}H_S\mathbf{b}$. **(vi)** Fix any $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$. Let $\mathbf{c} \in \mathbb{R}^n$ be the vector defined by

$$c_i = (\Sigma_{i\in N} a_i)/n = (\Sigma_{i\in N} b_i)/n \text{ for all } i \in N.$$

Hence we have by Claim 2(iv) that $\mathbf{c}H_I\mathbf{a}$ and $\mathbf{c}H_I\mathbf{b}$, which implies $\mathbf{a}H_I\mathbf{b}$. □

Appendix C

Example 1: Suppose that there are just two consumers, and the set of consumers is denoted by $N = \{1, 2\}$. There are m $(m \geq 2)$ types of consumption goods and the consumption set is $Y \equiv \mathbb{R}_+^m$ for all consumers. Denote $m^* = m - 1$. Setting the two utility functions U^1 and U^2 by $U^1(z_1, z_2, \cdots, z_m) = z_1\cdot(z_2\cdots z_m)^{1/m^*}$ and $U^2(z_1, z_2, \cdots, z_m) = z_1 + (\sqrt{z_2} + \cdots + \sqrt{z_m})/m^*$, define a profile $f = (\mathbf{p}^0, \mathbf{x}^0, \succsim)$ by $\mathbf{p}^0 = (1, 1/m^*, \cdots, 1/m^*)$, $\mathbf{x}^0 = (10, 10)$ and $\succsim_1$ and $\succsim_2$ on Y by $\mathbf{z} \succsim_i \mathbf{w} \Leftrightarrow U^i(\mathbf{z}) \geq U^i(\mathbf{w})$ for all $i \in N$ and all $\mathbf{z}, \mathbf{w} \in Y$. Then it holds that $f \in F$ and

$$d^1((1, q/m^*, \cdots, q/m^*), x: f) = (x/2, x/2q, \cdots, x/2q) \text{ for all } q, x > 0.$$

Setting $\mathbf{p} = (1, 1/m^*, \cdots, 1/m^*)$, $\mathbf{q} = (1, 3/m^*, \cdots, 3/m^*)$, $\mathbf{x} = (20, 20)$ and $\mathbf{y} = (30, 30)$, it holds that $(\mathbf{p}, \mathbf{x}), (\mathbf{q}, \mathbf{y}) \in A(f)$, $CV^1(\mathbf{p}, \mathbf{x}: f) = x_1 - \mu^1(\mathbf{p}, d^1(\mathbf{p}^0, x_1^0: f) : f) = 10$ and $CV^1(\mathbf{q}, \mathbf{y}: f) = y_1 - \mu^1(\mathbf{q}, d^1(\mathbf{p}^0, x_1^0: f) : f) = 30 - (10\sqrt{3}) > 12$, which implies that $CV^1(\mathbf{p}, \mathbf{x}: f) < CV^1(\mathbf{q}, \mathbf{y}: f)$. Moreover, it holds that $U^1(d^1(\mathbf{p}, x_1: f)) = 100 > U^1(d^1(\mathbf{q}, y_1: f)) = 75$. This means that $CV^1(\cdot, \cdot: f)$ does not represent $\succsim_1$ on $A(f)$. In case of the consumer 2, it holds that

$$d^2((1, q/m^*, \cdots, q/m^*), x : f) = (x - 1/4q,\ 1/4q^2, \cdots, 1/4q^2) \quad \text{for all } q > 0, x > 1/4q.$$

Setting $\mathbf{r} = (1, 5/m^*, \cdots, 5/m^*)$ and $\mathbf{w} = (20.3, 20.3)$, it holds that $(\mathbf{r}, \mathbf{w}) \in A(f)$, $d^2(\mathbf{p}, x_2: f) = (20 - 1/4, 1/4, \cdots, 1/4)$, $d^2(\mathbf{r}, w_2: f) = (20.3 - 0.05, 0.01, \cdots, 0.01)$, $CR^2(\mathbf{p}, \mathbf{x}: f) = x^2/x_2^0 = 2 > CR^2(\mathbf{r}, \mathbf{w}: f) = w^2/\mu^2(\mathbf{r}, d^2(\mathbf{p}^0, x_2^0: f) : f) = 1.99$, and that $U^2(d^2(\mathbf{r}, w_2: f)) = 20.35 > U^2(d^2(\mathbf{p}, x_2: f)) = 20.25$. This means that $CR^2(\cdot, \cdot: f)$ does not represent $\succsim_2$ on $A(f)$.

Example 2: Let $N = \{1, 2, \cdots, n\}$ be the set of all consumers, and let $Y \equiv \mathbb{R}^m_+$ be the consumption set. For all $i \in N$, let U^i be the utility function defined by

$$\begin{aligned} U^i(z_1, z_2, \cdots, z_m) &= -2z_1^{-1} - (\textstyle\sum_{j=2}^m z_j^{-2})/(m-1) \quad \text{if } i = 1, \\ &= -z_1^{-2} - 2(\textstyle\sum_{j=2}^m z_j^{-1})/(m-1) \quad \text{otherwise.} \end{aligned}$$

Define $\succsim \in \mathcal{M}^n$ by $\mathbf{z} \succsim_i \mathbf{w} \Leftrightarrow U^i(\mathbf{z}) \geq U^i(\mathbf{w})$ for all $i \in N$ and all $\mathbf{z}, \mathbf{w} \in Y$, and define a profile $f \in F$ by $f = (\mathbf{p}, \mathbf{x}, \succsim)$ where $\mathbf{p} = (1, 5/(m-1), \cdots, 5/(m-1))$ and $\mathbf{x} = (100, 103, 103, \cdots, 103)$. Setting $\mathbf{q} = (1, 1/(m-1), \cdots, 1/(m-1))$, $\mathbf{y} = (101, 102, 103, \cdots, 103)$, it holds that

$$\begin{aligned} &\textstyle\sum_{i\in N}\mu^i(\mathbf{p}, d^i(\mathbf{q}, x_i: f): f) - \sum_{i\in N}\mu^i(\mathbf{p}, d^i(\mathbf{q}, y_i: f): f) \\ &\quad = [\mu^1(\mathbf{p}, d^1(\mathbf{q}, 100: f): f) + \mu^2(\mathbf{p}, d^2(\mathbf{q}, 103: f): f)] - [\mu^1(\mathbf{p}, d^1(\mathbf{q}, 101: f): f) + \mu^2(\mathbf{p}, d^2(\mathbf{q}, 102: f): f)] \\ &\quad = (157{,}4228+422.5623) - (158.8212+418.2086) = 2.943\cdots > 0, \end{aligned}$$

$$\begin{aligned} &\textstyle\sum_{i\in N}\log\mu^i(\mathbf{p}, d^i(\mathbf{q}, x_i: f): f) - \sum_{i\in N}\log\mu^i(\mathbf{p}, d^i(\mathbf{q}, y_i: f): f) \\ &\quad = [\log\mu^1(\mathbf{p}, d^1(\mathbf{q}, 100: f): f) + \log\mu^2(\mathbf{p}, d^2(\mathbf{q}, 103: f): f)] \\ &\qquad\qquad - [\log\mu^1(\mathbf{p}, d^1(\mathbf{q}, 101: f): f) + \log\mu^2(\mathbf{p}, d^2(\mathbf{q}, 102: f): f)] \\ &\quad = (5.0589+6.0463) - (5.0678+6.0360) = 0.001\cdots > 0. \end{aligned}$$

Hence we have by (10) and (12) that $(\mathbf{q}, \mathbf{x})\ W_S^{EV}(f)\ (\mathbf{q}, \mathbf{y})$ and $(\mathbf{q}, \mathbf{x})\ W_S^{ER}(f)\ (\mathbf{q}, \mathbf{y})$. Since $\mathbf{y}$ is obtained from $\mathbf{x}$ by a progressive transfer under the fixed price vector $\mathbf{q}$, we have that the two social orderings $W^{EV}(f)$ and $W^{ER}(f)$ do not satisfy the Pigu-Dalton transfer principle for the alternatives.

On the other hand, setting $\mathbf{x}^* = (103, 100, 103, \cdots, 103)$ and $\mathbf{y}^* = (102, 101, 103, \cdots, 103)$, it holds that

$$\Sigma_{i\in N}\mu^i(\mathbf{p}, d^i(\mathbf{q}, x^*_i : f): f) - \Sigma_{i\in N}\mu^i(\mathbf{p}, d^i(\mathbf{q}, y^*_i : f): f)$$

$$= [\mu^1(\mathbf{p}, d^1(\mathbf{q}, 103: f): f) + \mu^2(\mathbf{p}, d^2(\mathbf{q}, 100: f): f)\,] - [\mu^1(\mathbf{p}, d^1(\mathbf{q}, 102: f): f) + \mu^2(\mathbf{p}, d^2(\mathbf{q}, 101: f): f)]$$

$$= (161.613+409.504) - (160.218+413.856) = -\ 2.957\cdots < 0,$$

$$\Sigma_{i\in N}\log \mu^i(\mathbf{p}, d^i(\mathbf{q}, x^*_i : f): f) - \Sigma_{i\in N}\log \mu^i(\mathbf{p}, d^i(\mathbf{q}, y^*_i : f): f)$$

$$= [\ \log \mu^1(\mathbf{p}, d^1(\mathbf{q}, 103: f): f) + \log \mu^2(\mathbf{p}, d^2(\mathbf{q}, 100: f): f)\ ]$$

$$-\ [\ \log \mu^1(\mathbf{p}, d^1(\mathbf{q}, 102: f): f) + \log \mu^2(\mathbf{p}, d^2(\mathbf{q}, 101: f): f)\ ]$$

$$= (5.0852+6.01495) - (5.07654+6.02552) = -\ 0.0019\cdots < 0.$$

Hence we have by (10) and (12) that $(\mathbf{q}, \mathbf{y}^*)\ W^{EV}_S(f)\ (\mathbf{q}, \mathbf{x}^*)$ and $(\mathbf{q}, \mathbf{y}^*)\ W^{ER}_S(f)\ (\mathbf{q}, \mathbf{x}^*)$. Since $\mathbf{y}^*$ is obtained from $\mathbf{x}^*$ by a progressive transfer under the fixed price vector $\mathbf{q}$, we have that the two social orderings $W^{EV}(f)$ and $W^{ER}(f)$ satisfy the Pigu-Dalton transfer principle for the alternatives.

Example 3: Let $N = \{ 1, 2, \cdots, n \}$ be the set of all consumers, and let $Y \equiv \mathbb{R}^m_+$ be the consumption set. Denote $m^* = m - 1$. Let U^1 be the utility function of the preference ordering $\succsim_1$ introduced in Example 1. Define $\boldsymbol{\succsim} \in \mathcal{M}^n$ by $\succsim_i = \succsim_1$ for all $i \in N$, and define a profile $f = (\mathbf{p}^0, \mathbf{x}^0, \boldsymbol{\succsim}) \in F$ where $\mathbf{p}^0 = (1, 1/m^*, \cdots, 1/m^*)$ and $\mathbf{x}^0 = (10, \cdots, 10)$. Moreover, set $\mathbf{p} = (1, 1/m^*, \cdots, 1/m^*)$, $\mathbf{q} = (1, 3/m^*, \cdots, 3/m^*)$, $\mathbf{x} = (20, \cdots, 20)$ and $\mathbf{y} = (30,\cdots, 30)$. Then it holds that $U^i(d^i(\mathbf{p}, x_i: f)) = 100 > U^1(d^1(\mathbf{q}, y_1: f)) = 75$ and $CV^i(\mathbf{q}, \mathbf{y}: f) > 12 > CV^i(\mathbf{p}, \mathbf{x}: f) = 10$. This means that $d^i(\mathbf{p}, x_i : f) \succ_i d^i(\mathbf{q}, y_i: f)$ for all $i \in N$ and $(\mathbf{q}, \mathbf{y})\ W^{CV}_S(f)\ (\mathbf{p}, \mathbf{x})$. Thus social ordering function W^{CV} does not satisfy the Pareto axiom. Let U^2 be the utility function of the preference ordering $\succsim_2$ introduced in Example 1. Define $\boldsymbol{\succsim}^* \in \mathcal{M}^n$ by $\succsim_i^* = \succsim_2$ for all $i \in N$, define a profile $g = (\mathbf{p}^0, \mathbf{x}^0, \boldsymbol{\succsim}^*) \in F$. Moreover, set $\mathbf{r} = (1, 5/m^*, \cdots, 5/m^*)$ and $\mathbf{w} = (20.3, \cdots, 20.3)$. Then it holds that $U^i(d^i(\mathbf{r}, w_i: g)) = 20.35 > U^i(d^i(\mathbf{p}, x_i : g)) = 20.25$ and $CR^i(\mathbf{p}, \mathbf{x}: f) = 2 > CR^i(\mathbf{r}, \mathbf{w}: f) = 1.99$ for all $i \in N$. This means that $d^i(\mathbf{r}, w_i : g) \succ_i d^i(\mathbf{p}, x_i: f)$ for all $i \in N$ and $(\mathbf{p}, \mathbf{x})\ W^{CR}_S(f)\ (\mathbf{r}, \mathbf{w})$. Thus social ordering function W^{CR} does not satisfy the Pareto axiom.

Example 4: Let $N = \{ 1, 2, \cdots, n \}$ be the set of all consumers, and let $Y \equiv \mathbb{R}^m_+$ be the consumption set. Denote $m^* = m - 1$. Let U^1 and U^2 be the utility functions of the preference orderings $\succsim_1$ and $\succsim_2$ introduced in Example 1, respectively. Define $\boldsymbol{\succsim}^* \in \mathcal{M}^n$ by $\succsim_1^* = \succsim_1$ and $\succsim_i^* = \succsim_2$ for all i

≥ 2, and define a profile $f = (\mathbf{p}^0, \mathbf{x}^0, \succsim^*) \in F$, where $\mathbf{p}^0 = (1, 1/m^*, \cdots, 1/m^*)$ and $\mathbf{x}^0 = (10, \cdots, 10)$. Set $\mathbf{p} = (1, 2/m^*, \cdots, 2/m^*)$, $\mathbf{x} = (20, 30, 30, \cdots, 30)$ and $\mathbf{y} = (30, 20, 30, \cdots, 30)$. Let θ be the permutation of N such that $\theta(1) = 2$, $\theta(2) = 1$, $\theta(i) = i$ for all $i \geq 3$. Then it holds that $\theta \circ \mathbf{x} = \mathbf{y}$. Then it holds that $d^1(\mathbf{p}^0, x_1^0 : f) = (5, 5, \cdots, 5)$, $d^1(\mathbf{p}, x_1 : f) = (10, 5, \cdots, 5)$, $d^1(\mathbf{p}, y_1 : f) = (15, 7.5, \cdots, 7.5)$, $EV^1(\mathbf{p}, \mathbf{x}: f) = \mu^1(\mathbf{p}^0, d^1(\mathbf{p}, x_1 : f) : f) - x_1^0 = 14.14 - 10 = 4.14$, and that

$$EV^1(\mathbf{p}, \mathbf{y}: f) = \mu^1(\mathbf{p}^0, d^1(\mathbf{p}, y_1 : f) : f) - x_1^0 = 11.23.$$

Moreover, it holds that $d^2(\mathbf{p}^0, x_2^0 : f) = (10 - 1/4, 1/4, \cdots, 1/4)$, $d^2(\mathbf{p}, x_2 : f) = (30 - 1/8, 1/16, \cdots, 1/16)$, $d^2(\mathbf{p}, y_2 : f) = (20 - 1/8, 1/16, \cdots, 1/16)$, $EV^2(\mathbf{p}, \mathbf{x}: f) = \mu^2(\mathbf{p}^0, d^2(\mathbf{p}, x_2 : f) : f) - x_2^0 = 19.875$, and that

$$EV^2(\mathbf{p}, \mathbf{y}: f) = \mu^2(\mathbf{p}^0, d^2(\mathbf{p}, y_2 : f) : f) - x_2^0 = 9.875.$$

Hence we have that $EV^1(\mathbf{p}, \mathbf{x}: f) + EV^2(\mathbf{p}, \mathbf{x}: f) = 24.015 > EV^1(\mathbf{p}, \mathbf{y}: f) + EV^2(\mathbf{p}, \mathbf{y}: f) = 21.105$, which implies $(\mathbf{p}, \mathbf{x})\ W_S^{EV}(f)\ (\mathbf{p}, \mathbf{y})$ and $(\mathbf{p}, \mathbf{x})\ W_S^{EV}(f)\ (\mathbf{p}, \theta \circ \mathbf{x})$. This means that the social ordering function W^{EV} does not satisfy the symmetry axiom. Moreover, it holds that $ER^1(\mathbf{p}, \mathbf{x}: f) = \mu^1(\mathbf{p}^0, d^1(\mathbf{p}, x_1 : f) : f) / x_1^0 = 1.414$, $ER^2(\mathbf{p}, \mathbf{x}: f) = \mu^2(\mathbf{p}^0, d^2(\mathbf{p}, x_2 : f) : f) / x_2^0 = 2.9875$, $ER^1(\mathbf{p}, \mathbf{y}: f) = \mu^1(\mathbf{p}^0, d^1(\mathbf{p}, y_1 : f) : f) / x_1^0 = 2.123$, $ER^2(\mathbf{p}, \mathbf{y}: f) = \mu^2(\mathbf{p}^0, d^2(\mathbf{p}, y_2 : f) : f) / x_2^0 = 1.9875$. Hence we have that $ER^1(\mathbf{p}, \mathbf{x}: f) \cdot ER^2(\mathbf{p}, \mathbf{x}: f) = 4.2243 > ER^1(\mathbf{p}, \mathbf{y}: f) \cdot ER^2(\mathbf{p}, \mathbf{y}: f) = 4.2194$, which implies $(\mathbf{p}, \mathbf{x})\ W_S^{ER}(f)\ (\mathbf{p}, \mathbf{y})$, and the social ordering function W^{ER} does not satisfy the symmetry axiom.

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