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North-South Asymmetry in Returns to Scale, Uneven Development, and the Population Puzzle

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Abstract

This paper develops a model of North-South trade and economic development to analyze how an increase in the growth rate of population affects the growth rate of real income per capita. We assume that the North is characterized by an increasing-returns-to-scale technology while the South is characterized by a decreasing-returns-to-scale technology. The main results are as follows: (i) an increase in the growth rate of population in the South decreases the growth rate of its own income per capita; (ii) a rise in the growth rate of population in the North either increases or decreases the growth rate of its own income per capita depending on conditions; (iii) population growth in one country raises the growth rate of income per capita in the other country; and (iv) even the decreasing-returns South can experience a positive growth rate of income per capita if a continuous improvement in the terms of trade is larger than a threshold value.

Keywords: North-South trade; economic development; population growth; non-scale growth; terms of trade

JEL Classification: F43; O11; O41

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1 Introduction

This paper attempts to give a probable explanation to what is called *the population puzzle* by combining a traditional model of North-South trade and economic development with a modern model of endogenous growth.[1)] The word "the population puzzle" is termed by Goodfriend and McDermott (1995), who point out that conclusions obtained from non-scale growth models, a class of endogenous growth models, are not consistent with empirical findings. Let us explain this issue below.

First, with respect to empirical findings, Goodfriend and McDermott (1995) refer to results of Kuznets (1973). Using postwar cross-country data for 63 countries, Kuznets (1973) shows that there is a statistically significant negative relationship between the growth rate of population and that of output per capita. Kuznets also shows that if the sample is divided into developed and developing countries, the correlation between the two growth rates becomes statistically insignificant. That is, the relationship between the growth of population and that of output per capita is not so unambiguous.

Second, we explain non-scale growth models. Since Jones (1995) challenged *the scale effects* of the endogenous growth model, non-scale growth models have gained attention in this field. In the *scale-growth* model, the growth rate of output per capita along the balanced growth path depends positively on the *size* of population. That is, the larger the size of population, the faster a country grows, which, however, seems counterfactual. Then, Jones (1995) attempts to remove the scale-effects and presents a *non-scale* growth model, from which the following conclusion is obtained: the growth rate of output per capita depends positively on the *rate* of population growth, and not on the size of population. That is, the higher the growth rate of population, the faster a country grows.[2)] However, this conclusion also seems inconsistent with the findings of Kuznets (1973) aside from the period considered. In this sense, Goodfriend and McDermott (1995) point out that population-driven models of growth must confront the population puzzle.

Existing non-scale growth models are built under a closed economy setting. However, the real world is an open economy, and accordingly conclusions obtained from closed economy models would not be applicable to the real world as they are. Therefore, we need an open economy model to apply the implication to the real world. In this respect, Christiaans

1) Traditional North-South models here mean a model that considers some asymmetries between developed and developing countries. See, for instance, Findlay (1980), Taylor (1981), Molana and Vines (1989), Darity (1990), and Sarkar (1997, 2001). Chui, Levine, Murshed, and Pearlman (2002) comprehensively survey the literature that combines growth and trade into models of North-South interaction. They deal with a combination of new trade and new (endogenous) growth models.

2) For a systematic exposition concerning scale effects and non-scale growth, see Jones (1999) and Christiaans (2004).

(2003) develops a small open economy model of non-scale growth that considers imported intermediate goods and exogenous export demand.[3)] The conclusion is that the relationship between the growth rate of population and that of output per capita differs depending on whether export goods of a country is price elastic or not. This model can be termed as *the modified non-scale growth model* in the sense that the unconditional positive correlation between the growth rate of population and that of output per capita is modified.[4)]

A model presented in this paper belongs to the class of modified non-scale growth models and is constructed under an open economy setting. Unlike Christiaans (2003), however, the model is a two-country model: two large countries engage in trade with each other. It is not appropriate that we apply the same specification to the developed and developing countries that have different structures. Thus, we develop a model of North-South trade and economic development that takes into account various asymmetries between the developed North and the developing South. Using it, we examine how the growth rate of population in each country affects the growth rates of income per capita in both countries.

Our model is based on Conway and Darity (1991), who develop a model of North-South trade and economic development along the lines of Kaldor's idea. Specifically, they model a situation where the North is characterized by increasing returns to scale while the South is characterized by decreasing returns to scale. Then, they analyze what consequences the asymmetry in returns to scale has for the growth rate of capital in each country, the terms of trade, and so forth. In this paper, we extend Conway and Darity's model in some respects. First, we assume that the Northern imports from the South are used for both consumption and intermediate inputs while the Southern imports from the North are used for both consumption and investment. In Conway and Darity's model, the Northern imports from the South are devoted only to consumption, which leads to the conclusion that the growth rate of capital stock in the North depends only on the Northern factors while the growth rate of capital stock in the South depends on both Northern and Southern factors.[5)] That is, there is no interdependence between the two growth rates. In our model, on the other hand, the North-South interdependence arises because the Southern good is used as input in the production of the Northern good. This idea is based on Dutt (1996). Second, we employ a

3) Christiaans (2003) introduces a learning-by-doing effect into Bardhan and Lewis's (1970) model to endogenize the rate of economic growth.

4) For modified non-scale growth models, see also Christiaans (2008) and Sasaki (2008). These papers deal with the relationship between trade specialization processes and industrialization under a small open economy setting.

5) Conway and Darity (1991) conclude that the growth rate of output in the North (South) is determined only by factors in the North (South). Note, however, that their definition of the long-run equilibrium is not correct and if the definition is corrected, our statement above is obtained. They define the long-run equilibrium as a situation in which capital-output ratio will be constant. However, when defining capital-output ratio in the South, they do not consider the terms-of-trade, which leads to the incorrect definition of capital-output ratio.

dynamic optimization technique to solve the model. In Conway and Darity's model, wage income is entirely consumed and profit income is entirely invested. In this paper, we do not differentiate workers from capitalists and treat the choice between consumption and saving as a dynamic optimization problem of a representative agent. Finally, we refer to other differences between Conway and Darity's and our models. In Conway and Darity's model, real wages in the short run are fixed both in the North and in the South. In the North, firms set the price with a mark-up on unit labor costs while in the South employment is determined by the equalization of the value of marginal products of labor and the nominal wage. In the long run of Conway and Darity's model, full employment is assumed and the real wages in both countries are adjusted so as to maintain full employment. In our model, in contrast, goods and factors markets in both countries are perfectly competitive and the marginal principle prevails both in the short run and in the long run.[6)]

The paper not only gives a probable explanation to the population puzzle, but also contributes to some issues in this field.

First, we contribute to the issue of the relationship between the terms of trade movement and economic development. Using a North-South model of endogenous growth, Felbermayr (2007) shows that even if the South specializes in a technologically stagnant sector, it can achieve a sustainable growth owing to endogenous, continuous improvements in the term-of-trade. Álvarez-Albelo and Perera-Tallo (2008) also reach a similar result using a different two-country endogenous growth model.[7)] As will be explained later, our model shows that if the rate of endogenous improvement in the Southern terms of trade is large, the growth rate of real income per capita in the South can be positive even if the Southern production is described by a decreasing-returns-to-scale technology. As a practical matter, there is an empirical observation that developing countries face a continuous improvement of their terms of trade. Felbermayr (2007), stated above, provides the following two observations: exports of developing countries to OECD countries are heavily biased toward consumption goods while their imports from OECD countries are mainly made up by investment goods; and the price of investment goods relative to consumption goods has been falling over time. Combining these two observations, we reach the conclusion that the terms of trade of developing countries continuously improves relative to developed countries.[8)]

Second, we contribute to the issue of two-country endogenous growth models. A large

6) The assumption that both agricultural and industrial markets are competitive and producers act as a price-taker is also adopted by Dutt (1992).

7) In their model, not a final good but a intermediate good is traded. One country is characterized by a Lucas-Rebelo type model while the other country is characterized by the neoclassical growth model. The terms of trade of the latter country permanently improves along the balanced growth path.

8) This phenomenon seems to contradict the Prebisch-Singer hypothesis. However, results of existing studies on this hypothesis are so diverse that the hypothesis is not necessarily an established proposition.

literature on two-country endogenous growth models focuses its attention on the balanced growth path and does not examine the dynamic stability. Ben-David and Loewy (2000), for example, develop a two-country endogenous growth model that considers knowledge as a factor of production and international spillovers of knowledge. In their model, each country specializes in production of a single good. They focus their analysis on the balanced growth path and do not examine the transitional dynamics. Felbermayr (2007) also confines the analysis to the balanced growth path equilibrium. Osang and Pereira (1997) present a two-country model of endogenous growth based on the Uzawa-Lucas human capital model. In this model as well as our model, the terms of trade between two countries continuously changes along the balanced growth path. However, they do not consider the dynamic stability. In contrast to these studies, we examine the dynamic stability and show the saddle-path stability of the model using a numerical method.

The remainder of the paper consists of the following five sections. Section 2 presents the basic framework of our model. Section 3 obtains the growth rates of endogenous variables on the balanced growth path. Section 4 derives the growth rate of income per capita in each country and investigates the relationship between population growth and income growth. Section 5 numerically shows the saddle-path stability of the long-run equilibrium. Section 6 concludes the paper.

2 The model

Consider a world that consists of the North, a developed country, and the South, a developing country. Each country completely specializes in production of a single good. The Northern good is used for consumption and investment in both countries. The Southern good, in contrast, is used for consumption in both countries and for intermediate input in the North. That is, we assume that the North exports a final consumption-cum-investment good to the South while the South exports a final consumption-cum-intermediate good to the North. Note that in the North value of total production differs from total value added due to the existence of imported intermediate input.

In this paper we focus on a competitive equilibrium path. As will be explained below, in the North, there exist externalities arising from capital accumulation. Therefore, a competitive equilibrium path diverges from an optimal path in which a social planner internalizes externalities. Steady states values of scale-adjusted variables are larger in the centrally planned case than in the decentralized case. However, growth rates on the balanced growth path are equal in both cases.

2.1 Firms

The North produces the N good according to the following Cobb-Douglas production function:

$$X_N = A_N K_N^{1-\alpha-\beta} L_N^{\alpha} M^{\beta}, \quad 0 < \alpha < 1, \ 0 < \beta < 1, \ \alpha + \beta < 1, \tag{1}$$

where X_N is total production, A_N external effects, K_N capital stock, L_N employment, and M imported intermediate goods. If A_N is regarded as an exogenous variable, the Northern production is constant returns to scale. Externalities due to capital accumulation are given by as follows:

$$A_N = K_N^{\alpha}. \tag{2}$$

We assume Marshallian externality in the following analysis. Accordingly, profit maximizing firms regard A_N as given exogenously. Substituting equation (2) into equation (1), we can rewrite the production function as follows:

$$X_N = K_N^{1-\beta} L_N^{\alpha} M^{\beta}, \tag{3}$$

which shows increasing returns to scale in K_N, L_N, and M.

The South produces the S good according to the following constant-returns-to-scale Cobb-Douglas production function:

$$Y_S = K_S^{1-a-b} L_S^{a} T^{b}, \quad 0 < a < 1, \ 0 < b < 1, \ a + b < 1, \tag{4}$$

where Y_S is output, K_S capital stock, L_S employment, and T land input. Suppose that supply of land is fixed. Then, we can normalized it to $T = 1$. From this, equation (4) can be rewritten as follows:

$$Y_S = K_S^{1-a-b} L_S^{a}, \tag{5}$$

which shows decreasing returns to scale in K_S and L_S.

Let $p \equiv p_S/p_N$ be the Southern terms of trade relative to the North with the N good being the numéraire. Then, profits of firms in the North and South are respectively given by $\Pi_N = X_N - r_N K_N - w_N L_N - pM$ and $\Pi_S = pY_S - r_S K_S - w_S L_S - qT$, where w_i denotes wage in country i, r_i rental rate of capital, and q rental rate of land: all prices are measured in terms of the N good. From profit maximizing conditions, we obtain the following relations:

$$\text{North}: w_N = \frac{\alpha X_N}{L_N}, \quad p = \frac{\beta X_N}{M}, \quad r_N = \frac{(1-\alpha-\beta)X_N}{K_N},$$
$$\text{South}: w_S = \frac{paY_S}{L_S}, \quad r_S = \frac{p(1-a-b)Y_S}{K_S}, \quad q = \frac{pbY_S}{T}.$$

Note that in the derivation above, A_N is treated as an exogenous variable by firms in the North.

2.2 Consumers

Consumers in both countries choose the flow of consumption to maximize the present discounted value of life time utility. We assume that instantaneous utility is given by the log-linear function.

$$U_N = \int_0^\infty [\gamma \ln C_N^S + (1-\gamma)\ln C_N^N] \exp(-\rho_N t)\, dt, \tag{6}$$
$$U_S = \int_0^\infty [\gamma \ln C_S^S + (1-\gamma)\ln C_S^N] \exp(-\rho_S t)\, dt, \tag{7}$$

where C_N^S, for instance, denotes the consumption of the S good in the North, γ a parameter governing an expenditure share for the S good, and ρ_i the rate of time preference, which can take a different value in each country.

The budget constraints are as follows.[9)]

$$\dot{K}_N = r_N K_N + w_N L_N - C_N^N - pC_N^S, \tag{8}$$
$$\dot{K}_S = r_S K_S + w_S L_S + qT - C_S^N - pC_S^S. \tag{9}$$

A dot over a variable denotes the time derivative of the variable: for example, $\dot{K}_N = dK_N(t)/dt$. For simplicity t is omitted unless it is needed. Since the total income in the North is given by $w_N L_N + r_N K_N = (1-\beta)X_N$ and that in the South is given by $w_S L_S + r_S K_S + qT = pY_S$, equations of motion for capital stock respectively lead to

$$\dot{K}_N = (1-\beta)X_N - C_N^N - pC_N^S, \tag{10}$$
$$\dot{K}_S = pY_S - C_S^N - pC_S^S. \tag{11}$$

9) We assume that a domestic asset market in each country is competitive but an international asset market does not exist. Therefore, in each country, the rental rate of capital and the rate of interest are equalized through asset arbitrage.

From equations (6), (7), (8), and (9), current-value Hamiltonian functions for the North and the South are respectively given by

$$\mathcal{H}_N = \gamma \ln C_N^S + (1-\gamma) \ln C_N^N + \lambda_N (r_N K_N + w_N L_N - C_N^N - p C_N^S), \tag{12}$$
$$\mathcal{H}_S = \gamma \ln C_S^S + (1-\gamma) \ln C_S^N + \lambda_S (r_S K_S + w_S L_S + qT - C_S^N - p C_S^S), \tag{13}$$

where λ_i is the co-state variable. The first-order necessary conditions are given by as follows:

$$\frac{\gamma}{C_N^S} - p\lambda_N = 0, \tag{14}$$
$$\frac{1-\gamma}{C_N^N} - \lambda_N = 0, \tag{15}$$
$$\lambda_N r_N = \rho_N \lambda_N - \dot{\lambda}_N, \tag{16}$$
$$\frac{\gamma}{C_S^S} - p\lambda_S = 0, \tag{17}$$
$$\frac{1-\gamma}{C_S^N} - \lambda_S = 0, \tag{18}$$
$$\lambda_S r_S = \rho_S \lambda_S - \dot{\lambda}_S. \tag{19}$$

In addition, we need the transversality condition.

$$\lim_{t\to+\infty} \lambda_i(t) K_i(t) \exp(-\rho_i t) = 0, \quad i = N, S. \tag{20}$$

From equations (16) and (19), we have $\dot{\lambda}_i(t)/\lambda_i(t) = \rho_i - r_i$, and consequently $\lambda_i(t) = \lambda_i(0) \exp\left[\int_0^t \{\rho_i - r_i(\tau)\}\, d\tau\right]$, which is substituted into equation (20).

$$\lim_{t\to+\infty} \lambda_i(0) K_i(t) \exp\left[-\int_0^t r_i(\tau)\, d\tau\right] = 0. \tag{21}$$

2.3 Market clearing conditions

Let us describe market clearing conditions for both goods. Taking into account the fact that investment in the South depends entirely on imports from the North while intermediate input in the North depends entirely on imports from the South, we can write market clearing conditions for both goods as follows:

$$X_N = C_N^N + C_S^N + I_N + I_S, \tag{22}$$
$$Y_S = C_N^S + C_S^S + M. \tag{23}$$

The trade balance condition is given by

$$pC_N^S + pM = C_S^N + I_S. \tag{24}$$

3 Balanced growth path

This section derives the balanced growth path (BGP). The BGP in the present paper is a situation where all variables grow at constant rates, which are not necessarily the same.

Using equations (15), (16), (18) and (19), we obtain two Euler equations for consumption. From equations (10) and (11), we obtain two equations of motion for capital stock. These four equations are presented below.[10)]

$$\frac{\dot{C}_N^N}{C_N^N} = (1-\alpha-\beta)\frac{X_N}{K_N} - \rho_N, \tag{25}$$

$$\frac{\dot{C}_S^N}{C_S^N} = (1-a-b)\frac{pY_S}{K_S} - \rho_S, \tag{26}$$

$$\frac{\dot{K}_N}{K_N} = (1-\beta)\frac{X_N}{K_N} - \frac{1}{1-\gamma}\frac{C_N^N}{K_N}, \tag{27}$$

$$\frac{\dot{K}_S}{K_S} = \frac{pY_S}{K_S} - \frac{1}{1-\gamma}\frac{C_S^N}{K_S}. \tag{28}$$

With these equations, let us derive the BGP growth rates of variables. In what follows we denote the growth rate of variable z as $g_z \equiv \dot{z}(t)/z(t)$.

To begin with, for consumption to grow at a constant rate, we need $g_{X_N} = g_{K_N}$ and $g_p + g_{Y_S} = g_{K_S}$ from equations (25) and (26). When $g_{X_N} = g_{K_N}$, the output-capital ratio in the North, X_N/K_N, becomes constant. Substituting $M = \beta X_N/p$ into the production function (1), we can rewrite the output-capital ratio as follows:

$$\frac{X_N}{K_N} = \beta^{\frac{\beta}{1-\beta}} L_N^{\frac{\alpha}{1-\beta}} p^{-\frac{\beta}{1-\beta}}. \tag{29}$$

Along the BGP, the right-hand side of equation (29) will be constant, that is, the rate of change in the right-hand side will be zero. Therefore, along the BGP, the following relation is obtained.

$$g_p^* = \frac{\alpha}{\beta} n_N > 0. \tag{30}$$

10) Euler equations for C_N^S and C_S^S are not needed directly because these two equations can be derived from the Euler equations for C_N^S and C_S^N with given p.

An asterisk ($*$) denotes the BGP value of a variable. Equation (30) implies that the terms of trade of the South (the North) improve (deteriorate) continuously along the BGP. In short, our model describes a situation where technical progress occurs in the North whereas does not occur in the South. Technical progress in the North decreases the relative price of the Northern good, so that the Southern terms of trade improve over time. Note that g_p^* is determined only by the parameters of the North and that the larger the population growth of the North, the higher the improvement rate of the South becomes. Let us focus on β. The parameter β corresponds to the ratio of imports of intermediate input to total production. Therefore, a large β implies a high degree of the Northern dependence on the Southern intermediate good. As the Northern dependence gets larger, the rate of improvement in the Southern terms of trade becomes smaller.

Next, for capital stock to grow at a constant rate, we require $g_{C_N^N} = g_{K_N}$ and $g_{C_S^N} = g_{K_S}$ from equations (27) and (28) provided that the output-capital ratios in both countries are constant. In this case, C_N^N and C_S^N must grow at the same rate; otherwise, world consumption for the N good does not grow at a constant rate. From this observation we have $g_{C_N^N} = g_{C_S^N}$, so that $g_{K_N} = g_{K_S}$, that is, along the BGP capital stocks in both countries grow at the same rate. Applying $g_{K_S} = g_{K_N} = g_{X_N}$ to $g_p + g_{Y_S} = g_{K_S}$, we obtain $g_p^* + g_{Y_S}^* = g_{X_N}^*$ along the BGP. Using these results for the production functions in both countries, we finally get the growth rate of capital stock in each country as follows:

$$g_{K_N}^* = g_{K_S}^* = \phi n_N + \psi n_S, \quad \text{where } \phi \equiv \frac{\alpha}{\beta(a+b)},\ \psi \equiv \frac{a}{a+b}. \tag{31}$$

These growth rates depend on the growth rates of population and the parameters of the production functions. Using (31), we can obtain the growth rates of other endogenous variables.

Finally, let us demonstrate that the transversality condition (21) holds. As has been shown above, along the BGP, $g_{K_N}^* = g_{K_S}^* = \phi n_N + \psi n_S$ holds. After some calculations, the BGP rental rates of capital in both countries are given by $r_N^* = \rho_N + \phi n_N + \psi n_S$ and $r_S^* = \rho_S + \phi n_N + \psi n_S$. From these observations we obtain $r_N^* - g_{K_N}^* = \rho_N > 0$ and $r_S^* - g_{K_S}^* = \rho_S > 0$. Applying these results to equation (21), we can see that the transversality condition holds.

4 Population growth and the growth of income per capita

We here focus on the BGP growth rate of real income per capita, $g_{y_i}^*$ $(i = N, S)$. For this purpose we have to obtain real national income, which in turn requires an appropriate definition of the consumption price index.

Let p_C denote the price index that is consistent with the expenditure minimizing problem.

Then, the price index is given by $p_C = \gamma^{-\gamma}(1-\gamma)^{-(1-\gamma)} p_N^{1-\gamma} p_S^{\gamma} = \gamma^{-\gamma}(1-\gamma)^{-(1-\gamma)} p^{\gamma}$. Since both countries face the common relative price, p, and preferences are same, the price index is also same in both countries. Note that the relation $g_{p_C} = \gamma g_p$ holds between the price index and the terms of trade.

From the analysis above, nominal national incomes in the North and the South are $X_N - pM$ and pY_S, respectively. Both are measured in terms of the N good. Accordingly, real national incomes in the North and the South are respectively given by $(X_N - pM)/p_C$ and pY_S/p_C. Using these real national incomes, we can compute the growth rates of real incomes per capita along the BGP as follows:

$$g^*_{y_N} = \frac{\alpha[1-\gamma(a+b)] - \beta(a+b)}{\beta(a+b)} n_N + \frac{a}{a+b} n_S, \tag{32}$$

$$g^*_{y_S} = \frac{\alpha[1-\gamma(a+b)]}{\beta(a+b)} n_N - \frac{b}{a+b} n_S. \tag{33}$$

Note that when $n_N = n_S$, we have $g^*_{y_N} = g^*_{y_S}$, which suggests that the difference between $g^*_{y_N}$ and $g^*_{y_S}$ arises from the difference between n_N and n_S. Therefore, the growth rates of real national incomes are identical.[11] From equations (32) and (33), we obtain the following propositions:[12]

Proposition 1

(i) A rise in the rate of population growth in the North increases or decreases the BGP growth rate of its own real income per capita according as $\frac{\alpha}{\beta} \gtrless \frac{a+b}{1-\gamma(a+b)}$.

(ii) A rise in the rate of population growth in the South decreases the BGP growth rate of its own real income per capita.

(iii) A rise in the rate of population growth in one country leads to a rise in the BGP growth rate of real income per capita in the other country.

(iv) Provided that the BGP rate of improvement in the Southern terms of trade is large, the BGP growth rate of real income per capita in the South can be positive even if the Southern production is described by a decreasing-returns-to-scale technology.

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11) Note that the growth rate of real national income is equal to that of aggregate consumption. Aggregate consumption is defined as $C_N \equiv (C_N^N)^{1-\gamma}(C_N^S)^{\gamma}$ in the North and $C_S \equiv (C_S^N)^{1-\gamma}(C_S^S)^{\gamma}$ in the South. Let E_N and E_S be the nominal expenditure in the North and that in the South, respectively. Then, we have $E_N = p_C C_N$ and $E_S = p_C C_S$.

12) It is possible that the growth rate of real income per capita in the North becomes negative depending on parametric conditions. As will be shown in section **5.2**, however, our numerical examples show that we probably have $g^*_{y_N} > 0$.

(v) *Even if a country's own rate of population growth is zero, the BGP growth rate of real income per capita of the country can be positive.*

Propositions 1-(ii) and 1-(iii) are easily checked by observing the coefficients of n_N and n_S in equations (32) and (33).

On Proposition 1-(i): The coefficient of n_N in $g^*_{y_N}$ is positive or negative according as $\alpha[1-\gamma(a+b)]-\beta(a+b) \gtrless 0$, which is equivalent to the condition in Proposition 1-(i). Given the parameters of the South, the larger α and/or the smaller β, the larger α/β becomes, so that the coefficient of n_N in $g^*_{y_N}$ is likely to become positive. As has been stated above, α measures the degree of externalities due to capital accumulation and β measures the degree of the Northern dependence on the Southern intermediate good.

On Proposition 1-(iv): Rearranging the condition $g^*_{y_S} \gtrless 0$ and using $g^*_p = \frac{\alpha}{\beta} n_N$ along the BGP, we obtain the following relation:

$$g^*_p \gtrless \frac{bn_S}{1-\gamma(a+b)}. \tag{34}$$

This means that if the rate of improvement in the Southern terms of trade is larger than the right-hand side, then the growth rate of real income per capita is positive.

On Proposition 1-(v): Substituting $n_N = 0$ and $n_S = 0$ into $g^*_{y_N}$ and $g^*_{y_S}$, respectively, we have

$$g^*_{y_N} = \frac{a}{a+b} n_S > 0, \tag{35}$$

$$g^*_{y_S} = \frac{\alpha[1-\gamma(a+b)]}{\beta(a+b)} n_N > 0. \tag{36}$$

In this case, the Northern growth rate is determined only by the Southern factors while the Southern growth rate is determined only by the Northern factors. In existing nons-scale growth models for closed economy, the growth rate of real income per capita will be zero when population growth is zero. In our model, however, the growth rate of income per capita is positive in each country due to the presence of international trade.

Here, referring to the existing literature, we add an explanation to the real national income used in the present paper. Temple (2005) points out that in calculating real national income in an open economy setting, there is an important distinction between a GDP price index and a cost-of-living index because the structure of consumption and that of production can be different. When measuring economic welfare, we have to use the cost-of-living index, which is nothing less than the price index p_C in our model. This distinction corresponds to the difference between real GDP (gross domestic product) and real GDI (gross domestic income) in the National Accounts. Real GDI is equal to real GDP plus the trading gains (or less trading losses) resulting from changes in the terms of trade. The importance of this distinction is emphasized by Álvarez-Albelo and Perera-Tallo (2007) and Kohli (2004). Let us consider this issue in our model.

First, national income measured in terms of the N good is given by $(1-\beta)X_N$, pY_S, and the growth rate of the national income becomes g_{X_N}, $g_p + g_{Y_S}$. Second, national income measured in terms of the good produced in each country is given by $(1-\beta)X_N$, Y_S, and the growth rate of the national income becomes g_{X_N}, g_{Y_S}. Third, national income measured in terms of the cost-of-living index, that is, our real national income (real GDI) is given by $(1-\beta)X_N/p_C$, pY_S/p_C, and the growth rate of the national income becomes $g^*_{X_N} - \gamma g^*_p$, $g^*_{Y_S} + (1-\gamma)g^*_p$.

In the first measure, the growth rates of both countries are equal because $g^*_{X_N} = g^*_p + g^*_{Y_S}$ along the BGP. In the second measure, with $g^*_p > 0$, we have $g^*_{X_N} > g^*_{Y_S}$. In the third measure, the growth rates of both countries are equal because $g^*_{X_N} = g^*_p + g^*_{Y_S}$. From this we can see that the first measure, compared with the third measure, overestimates the growth rates of both countries by γg^*_p and that the second measure, compared with the third measure, overestimates the growth rate of the North by γg^*_p and underestimates the growth rate of the South by $(1-\gamma)g^*_p$.

5 Existence and stability of the long-run equilibrium

This section investigates the existence and the local stability of a steady state. Unless there exists a path converging to the steady state, analysis along the BGP will lose its importance. However, little literature on two-country endogenous growth models has dealt with this issue. Farmer and Lahiri (2005), for instance, investigate multiple equilibria and indeterminacy using a two-country endogenous growth model. In their model, however, a homogeneous good is produced in both countries, which implies that the terms of trade between two countries are always unity. A lot of existing studies confine their analysis to the BGP. In the first place, there exists little literature that investigates the dynamic stability of two-country neoclassical growth models. Brecher, Chen, and Choudhri (2005) is one of the few studies that exist. The main reason for this is that in the two-country setting, we have a lot of endogenous variables, so that we have to analyze a system of differential equations of high order. From our model we obtain a system of four differential equations. Analytical treatment of the system is troublesome, and accordingly numerical simulation is used.[13)]

13) Arnold (2006) analytically proves that there exists a unique path converging to a steady state (saddle-path stability) in Jones's (1995) model, which produces a system of four differential equations.

5.1 Existence of the equilibrium

Because K_N, K_S, C_N^N, and C_S^N keep on growing along the BGP, we cannot investigate the dynamic stability of the system without modifications. For this reason, we introduce the following scale-adjusted variables with consideration for the BGP growth rates of the above-mentioned variables and the terms of trade:[14)]

$$\pi \equiv \frac{p}{L_N^{\frac{\alpha}{\beta}}},\ k_N \equiv \frac{K_N}{L_N^{\phi} L_S^{\psi}},\ k_S \equiv \frac{K_S}{L_N^{\phi} L_S^{\psi}},\ c_N^N \equiv \frac{C_N^N}{L_N^{\phi} L_S^{\psi}},\ c_S^N \equiv \frac{C_S^N}{L_N^{\phi} L_S^{\psi}}. \tag{37}$$

[Figure 1 to be inserted here]

Here, we show that the newly defined terms of trade, π, can be expressed as a function of the other scale-adjusted variables. Expressing the market clearing condition for the S good (23) by the scale-adjusted variables, we obtain the following modified market clearing condition:

$$\pi k_S^{1-a-b} - \frac{\gamma}{1-\gamma}(c_N^N + c_S^N) = \beta^{\frac{1}{1-\beta}} \pi^{-\frac{\beta}{1-\beta}} k_N, \tag{38}$$

where c_N^S and c_S^S are transformed into c_N^N and c_S^N, respectively, with the use of equations (14), (15), (17), and (18). In equation (38), the left-hand side (LHS) and right-hand side (RHS) can be regarded as functions of π, which are drawn in Figure 1. As Figure 1 shows, the intersection of LHS and RHS uniquely determines π. Therefore, the scale-adjusted terms of trade π lead to a function of the other scale-adjusted variables.

$$\pi = \pi(\underset{+}{k_N}, \underset{-}{k_S}, \underset{+}{c_N^N}, \underset{+}{c_S^N}), \tag{39}$$

where a sign below a variable show the sign of the corresponding partial derivative of the variable.[15)]

14) See Eicher and Turnovsky (1999, 2001) for a technique based on scale-adjusted variables.

15) Specifically, each partial derivative is given by as follows:

$$\frac{\partial \pi}{\partial k_N} = \frac{\beta^{\frac{1}{1-\beta}} \pi^{-\frac{\beta}{1-\beta}}}{\Delta} > 0,\ \frac{\partial \pi}{\partial k_S} = -\frac{(1-a-b)\pi k_S^{-a-b}}{\Delta} < 0,\ \frac{\partial \pi}{\partial c_N^N} = \frac{\partial \pi}{\partial c_S^N} = \frac{\frac{\gamma}{1-\gamma}}{\Delta} > 0,$$

where

$$\Delta \equiv k_S^{1-a-b} + \frac{\beta^{\frac{2-\beta}{1-\beta}}}{1-\beta} \pi^{-\frac{1}{1-\beta}} k_N > 0.$$

Substituting equation (37) into equations (25)–(28), we obtain the following differential equations for the scale-adjusted variables:

$$\dot{k}_N = k_N \cdot \left[(1-\beta)\beta^{\frac{\beta}{1-\beta}}\pi^{-\frac{\beta}{1-\beta}} - \frac{\gamma}{1-\gamma}\frac{c_N^N}{k_N} - \phi n_N - \psi n_S \right], \tag{40}$$

$$\dot{k}_S = k_S \cdot \left[\pi k_S^{-a-b} - \frac{\gamma}{1-\gamma}\frac{c_S^N}{k_S} - \phi n_N - \psi n_S \right], \tag{41}$$

$$\dot{c}_N^N = c_N^N \cdot \left[(1-\alpha-\beta)\beta^{\frac{\beta}{1-\beta}}\pi^{-\frac{\beta}{1-\beta}} - \rho_N - \phi n_N - \psi n_S \right], \tag{42}$$

$$\dot{c}_S^N = c_S^N \cdot \left[(1-a-b)\pi k_S^{-a-b} - \rho_S - \phi n_N - \psi n_S \right]. \tag{43}$$

The steady state to the above system is a situation where $\dot{k}_N = \dot{k}_S = \dot{c}_N^N = \dot{c}_S^N = 0$, from which the following steady state values are uniquely determined:

$$\pi^* = \beta \left(\frac{\rho_N + \phi n_N + \psi n_S}{1-\alpha-\beta} \right)^{-\frac{1-\beta}{\beta}}, \tag{44}$$

$$k_S^* = \left[\frac{\rho_S + \phi n_N + \psi n_S}{(1-a-b)\pi^*} \right]^{-\frac{1}{a+b}}, \tag{45}$$

$$c_S^{N*} = \frac{1-\gamma}{\gamma} \cdot \frac{\rho_S + (a+b)(\phi n_N + \psi n_S)}{1-a-b} \cdot k_S^*, \tag{46}$$

$$k_N^* = \frac{(1-\alpha-\beta)(\phi n_N + \psi n_S)}{\rho_N + (\alpha+\beta)(\phi n_N + \psi n_S)} \cdot k_S^*, \tag{47}$$

$$c_N^{N*} = \frac{1-\gamma}{\gamma} \cdot \frac{(1-\beta)\rho_N + \alpha(\phi n_N + \psi n_S)}{1-\alpha-\beta} \cdot k_N^*. \tag{48}$$

Since $1-\alpha-\beta > 0$ and $1-a-b > 0$, all steady state values are positive.

5.2 Stability of the equilibrium

In our system of differential equations, c_N^N and c_S^N are jump variables while k_N and k_S are predetermined variables. Therefore, if the Jacobian matrix corresponding to the system of differential equations has two positive eigenvalues and two negative eigenvalues, then the steady state obtained above is saddle-path stable.[16)] Here, we refer to both positive (negative) real eigenvalues and complex eigenvalues with positive (negative) real parts as *positive (negative) eigenvalues*. To know signs of eigenvalues analytically is very difficult,

16) If the number of jump variables are equal to that of positive eigenvalues, a system of differential equations is saddle-path stable. If the number of positive eigenvalues exceeds that of jump variables, the system is unstable. If, in contrast, the number of jump variables exceeds that of positive eigenvalues, the system is in indeterminacy.

and consequently we resort to a numerical simulation.[17)] Let **J** be the Jacobian matrix of our system. Then, **J** is a 4×4 matrix, and each element is presented in Appendix A.

We confine our numerical analysis to the following four cases: Cases 1, 2, 3, and 4. These four cases are chosen to produce Proposition 1 obtained above and shown in Table 1: $g^*_{y_N} > 0$ (mentioned in footnote 12), $g^*_{y_S} \gtrless 0$ (Proposition 1-(iv)), and $\partial g^*_{y_N}/\partial n_N \gtrless 0$ (Proposition 1-(i)). In what follows, we present combinations of parameters corresponding to these four cases, and confirm whether there exists a unique path converging to the steady state. Table 2 shows a combination of parameters.

[Tables 1 and 2 to be inserted here]

Table 3 presents results of numerical analysis based on the values of the parameters in Table 2. Here, $\chi_1, \ldots, \chi_4$ denote eigenvalues. In every case we have two positive eigenvalues and two negative eigenvalues. Therefore, the steady state is characterized by saddle-path stability.

[Table 3 to be inserted here]

Let us demonstrate how the scale-adjusted capital stock in each country converges to the steady state value with an initial value being given. To this end, we need to solve the system of differential equations using the eigenvalues and eigenvectors thus obtained. This procedure is shown in Appendix B. For simplicity, let the initial values be $K_N(0) = K_S(0) = 1$ and $L_N(0) = L_S(0) = 1$. Then, the initial values of scale-adjusted capital stock are also $k_N(0) = k_S(0) = 1$. Case 2 is adopted for example. Figures 2 and 3 show the dynamics of k_N and k_S, respectively: the horizontal axis measures t while the vertical axis denotes k_N, k_S. Both figures show that starting from the initial value the scale-adjusted capital stock converges to the steady state value. In Case 2, we have $k^*_N = 16.2936$ and $k^*_S = 64.8417$, which might be counterintuitive because the capital stock in the developed North is smaller than that in the developing South. However, k_N and k_S are scale-adjusted capital stock, and not capital stock per capita. When, for example, $t = 1000$, the capital stock per capita in the North and that in the South are given by $K_N/L_N = 9.86 \times 10^{27}$ and $K_S/L_S = 1.78 \times 10^{24}$, respectively, which clearly shows that $K_N/L_N > K_S/L_S$.

[Figures 2 and 3 to be inserted here]

17) For numerical calculations, we use *Mathematica* 4 of Wolfram Research Inc.

6 Concluding remarks

In this paper we have united a traditional model of North-South trade and economic development and a modern model of endogenous growth. Using the model, we have investigated how a rise in the growth rate of population affects the growth rate of real income per capita in each country. The following are the main results: (i) an increase in the growth rate of population in the South decreases the growth rate of its own income per capita; (ii) a rise in the growth rate of population in the North either increases or decreases the growth rate of its own income per capita depending on conditions; (iii) population growth in one country raises the growth rate of income per capita in the other country; and (iv) even the decreasing-returns South can experience a positive growth rate of income per capita if a continuous improvement in the terms of trade is larger than a threshold value. As these results show, in developed countries the correlation between the growth rate of population and that of income per capita can be either positive or negative while in developing countries the correlation is negative. Therefore, empirical analysis lumping developed and developing countries together will yield an ambiguous correlation between the two growth rates.

Let us add an explanation to the fourth result. It is true that the South can experience a sustainable growth in per capita income depending on conditions. Even in this case, however, the growth rate of income per capita in the South is necessarily lower than that in the North if the growth rate of population in the South is higher than that in the North. For this reason, if the initial level of income per capita in the South is lower than that in the North, the South cannot catch up with the North in its income level. Therefore, uneven development still remains though the Southern terms of trade continue to improve.

Our model does not consider international capital mobility. Along the BGP, if the rates of time preference are equal in both countries, the rental rates of capital are equalized, that is, $r_N^* = r_S^*$, so that capital does not move between the two countries. Off the BGP, however, we have $r_N^* \neq r_S^*$ even if $\rho_N = \rho_S$, and consequently capital can move between the countries in search of higher returns. The rental rates of capital off the BGP depend on the scale-adjusted terms of trade, $\pi(t)$. Therefore, in order to investigate capital movement, we have to examine the dynamics of $\pi(t)$ in detail. Analytical treatment of the capital movement will be very difficult because the dynamics of $\pi(t)$ are implicitly determined by the dynamics of the other variables, k_N, k_S, c_N^N, and c_S^N. In this case, numerical simulation will be useful.

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Appendix A

The following are elements of the Jacobian matrix evaluated the BGP:

$$J_{11} = \frac{\partial \dot{k}_N}{\partial k_N} = k_N \left[-\beta^{\frac{1}{1-\beta}} \pi^{-\frac{1}{1-\beta}} \frac{\partial \pi}{\partial k_N} + \frac{\gamma}{1-\gamma} \frac{c_N^N}{k_N^2} \right], \tag{49}$$

$$J_{12} = \frac{\partial \dot{k}_N}{\partial k_S} = k_N \left[-\beta^{\frac{1}{1-\beta}} \pi^{-\frac{1}{1-\beta}} \frac{\partial \pi}{\partial k_S} \right] > 0, \tag{50}$$

$$J_{13} = \frac{\partial \dot{k}_N}{\partial c_N^N} = k_N \left[-\beta^{\frac{1}{1-\beta}} \pi^{-\frac{1}{1-\beta}} \frac{\partial \pi}{\partial c_N^N} - \frac{\gamma}{1-\gamma} \frac{1}{k_N} \right] < 0, \tag{51}$$

$$J_{14} = \frac{\partial \dot{k}_N}{\partial c_S^N} = k_N \left[-\beta^{\frac{1}{1-\beta}} \pi^{-\frac{1}{1-\beta}} \frac{\partial \pi}{\partial c_S^N} \right] < 0, \tag{52}$$

$$J_{21} = \frac{\partial \dot{k}_S}{\partial k_N} = k_S \left[k_S^{-a-b} \frac{\partial \pi}{\partial k_N} \right] > 0, \tag{53}$$

$$J_{22} = \frac{\partial \dot{k}_S}{\partial k_S} = k_S \left[(-a-b) k_S^{-a-b-1} \pi + k_S^{-a-b} \frac{\partial \pi}{\partial k_S} + \frac{\gamma}{1-\gamma} \frac{c_S^N}{k_S^2} \right], \tag{54}$$

$$J_{23} = \frac{\partial \dot{k}_S}{\partial c_N^N} = k_S \left[k_S^{-a-b} \frac{\partial \pi}{\partial c_N^N} \right] > 0, \tag{55}$$

$$J_{24} = \frac{\partial \dot{k}_S}{\partial c_S^N} = k_S \left[k_S^{-a-b} \frac{\partial \pi}{\partial c_S^N} - \frac{\gamma}{1-\gamma} \frac{1}{k_S} \right], \tag{56}$$

$$J_{31} = \frac{\partial \dot{c}_N^N}{\partial k_N} = c_N^N \left[(1-\alpha-\beta) \beta^{\frac{\beta}{1-\beta}} \left(-\frac{\beta}{1-\beta} \right) \pi^{\frac{\beta-2}{1-\beta}} \frac{\partial \pi}{\partial k_N} \right] < 0, \tag{57}$$

$$J_{32} = \frac{\partial \dot{c}_N^N}{\partial k_S} = c_N^N \left[(1-\alpha-\beta) \beta^{\frac{\beta}{1-\beta}} \left(-\frac{\beta}{1-\beta} \right) \pi^{\frac{\beta-2}{1-\beta}} \frac{\partial \pi}{\partial k_S} \right] > 0, \tag{58}$$

$$J_{33} = \frac{\partial \dot{c}_N^N}{\partial c_N^N} = c_N^N \left[(1-\alpha-\beta) \beta^{\frac{\beta}{1-\beta}} \left(-\frac{\beta}{1-\beta} \right) \pi^{\frac{\beta-2}{1-\beta}} \frac{\partial \pi}{\partial c_N^N} \right] < 0, \tag{59}$$

$$J_{34} = \frac{\partial \dot{c}_N^N}{\partial c_S^N} = c_N^N \left[(1-\alpha-\beta) \beta^{\frac{\beta}{1-\beta}} \left(-\frac{\beta}{1-\beta} \right) \pi^{\frac{\beta-2}{1-\beta}} \frac{\partial \pi}{\partial c_S^N} \right] < 0, \tag{60}$$

$$J_{41} = \frac{\partial \dot{c}_S^N}{\partial k_N} = c_S^N \left[(1-a-b) k_S^{-a-b} \frac{\partial \pi}{\partial k_N} \right] > 0, \tag{61}$$

$$J_{42} = \frac{\partial \dot{c}_S^N}{\partial k_S} = c_S^N \left[(-a-b)(1-a-b) \pi k_S^{-a-b-1} + (1-a-b) k_S^{-a-b} \frac{\partial \pi}{\partial k_S} \right] < 0, \tag{62}$$

$$J_{43} = \frac{\partial \dot{c}_S^N}{\partial c_N^N} = c_S^N \left[(1-a-b) k_S^{-a-b} \frac{\partial \pi}{\partial c_N^N} \right] > 0, \tag{63}$$

$$J_{44} = \frac{\partial \dot{c}_S^N}{\partial c_S^N} = c_S^N \left[(1-a-b) k_S^{-a-b} \frac{\partial \pi}{\partial c_S^N} \right] > 0. \tag{64}$$

Appendix B

We solve the system of differential equations:

$$\begin{pmatrix} k_N(t) - k_N^* \\ k_S(t) - k_S^* \\ c_N^N(t) - c_N^{N*} \\ c_S^N(t) - c_S^{N*} \end{pmatrix} = B_1 e^{\chi_1 t} \begin{pmatrix} b_{11} \\ b_{21} \\ b_{31} \\ b_{41} \end{pmatrix} + B_2 e^{\chi_2 t} \begin{pmatrix} b_{12} \\ b_{22} \\ b_{32} \\ b_{42} \end{pmatrix}, \tag{65}$$

where χ_1 and χ_2 denote negative eigenvalues, $b_{11}, \ldots, b_{41}$ eigenvectors corresponding to χ_1, and $b_{12}, \ldots, b_{42}$ eigenvectors corresponding to χ_2. B_1 and B_2 are constants which should be determined by initial conditions.

In our model, the initial values of k_N and k_S are historically given but those of c_N^N and c_S^N are determined so that they will be located on the saddle path. Let us show the procedure in the following. At $t = 0$, the following relations hold:

$$\begin{pmatrix} k_N(0) - k_N^* \\ k_S(0) - k_S^* \\ c_N^N(0) - c_N^{N*} \\ c_S^N(0) - c_S^{N*} \end{pmatrix} = B_1 \begin{pmatrix} b_{11} \\ b_{21} \\ b_{31} \\ b_{41} \end{pmatrix} + B_2 \begin{pmatrix} b_{12} \\ b_{22} \\ b_{32} \\ b_{42} \end{pmatrix}. \tag{66}$$

Here, $k_N(0)$ and $k_S(0)$ are given exogenously and k_N^*, k_S^*, b_{11}, b_{12}, b_{21}, and b_{22} are also known to us. From this, the two upper equations in equation (66) constitute a system of equations with respect to B_1 and B_2:

$$k_N(0) - k_N^* = B_1 b_{11} + B_2 b_{12}, \tag{67}$$

$$k_S(0) - k_S^* = B_1 b_{21} + B_2 b_{22}. \tag{68}$$

Solving the equations above, we can determine B_1 and B_2.

Substituting B_1 and B_2 obtained now into the two lower equations in equation (66) and considering that c_N^{N*}, c_S^{N*}, b_{31}, b_{32}, b_{41}, and b_{42} are already known, we obtain the system of equations with respect to $c_N^N(0)$ and $c_S^N(0)$:

$$c_N^N(0) - c_N^{N*} = B_1 b_{31} + B_2 b_{32}, \tag{69}$$

$$c_S^N(0) - c_S^{N*} = B_1 b_{41} + B_2 b_{42} \tag{70}$$

From this, $c_N^N(0)$ and $c_S^N(0)$ are determined.

Figures and Tables

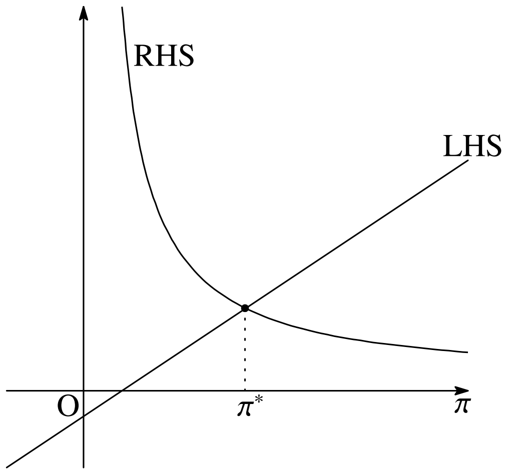

Figure 1: Determination of the scale-adjusted terms of trade

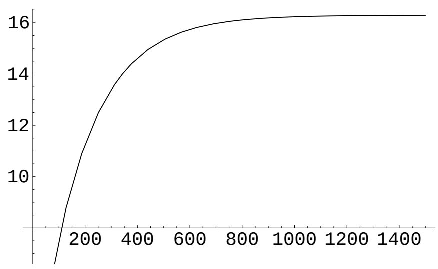

Figure 2: Dynamics of k_N

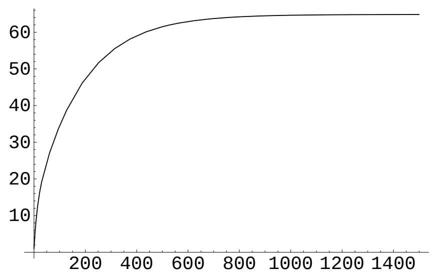

Figure 3: Dynamics of k_S

Table 1: Four possible combinations of $g^*_{y_N}$, $g^*_{y_S}$, and $\partial g^*_{y_N}/\partial n_N$

	$g^*_{y_N}$	$g^*_{y_S}$	$\partial g^*_{y_N}/\partial n_N$
Case 1	+	−	−
Case 2	+	+	+
Case 3	+	+	−
Case 4	+	−	+

Table 2: Numerical values of the parameters

	$\gamma = 2/5,\ \rho_N = 1/50,\ \rho_S = 1/25,\ n_N = 1/50,\ n_S = 3/100$
Case 1	$a = 1/3,\ b = 1/3,\ \alpha = 1/3,\ \beta = 1/2$
Case 2	$a = 1/3,\ b = 1/3,\ \alpha = 1/2,\ \beta = 1/4$
	$\gamma = 1/5,\ \rho_N = 1/50,\ \rho_S = 1/25,\ n_N = 1/100,\ n_S = 1/50$
Case 3	$a = 1/2,\ b = 1/4,\ \alpha = 1/4,\ \beta = 1/3$
Case 4	$a = 1/4,\ b = 1/2,\ \alpha = 1/3,\ \beta = 1/3$

Table 3: Eigenvalues of the Jacobian matrix in Cases 1–4

	χ_1	χ_2	χ_3	χ_4
Case 1	$-0.024 - 0.014i$	$-0.024 + 0.014i$	0.064	0.13
Case 2	−0.074	−0.0047	0.089	0.21
Case 3	−0.023	−0.00058	0.036	0.066
Case 4	−0.020	−0.00069	0.045	0.069